Detection and continuation of a homoclinic cycle–to–cycle connection in a food chain model.

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Abstract

Accompanying manuscript to the demonstration of the detection and continuation of a homoclinic cycle-to-cycle connection in the 3D food chain model by Rosenzweig–MacArthur with the bifurcation software package AUTO, by use of the homotopy method described in the paper. The files are downloadable from http://www.bio.vu.nl/thb/research/project/globif.

1 Disclaimer

The following results have been obtained under Sun Solaris 8, using a FORTRAN *compiler f77 for* AUTO97*, and using a* FORTRAN *compiler f95 for* AUTO07P*. The results might differ slighthy using a different compiler or a different version of* AUTO*.*

2 Introduction

The scaled Rosenzweig-MacArthur system (1963) is a three-level food chain model from theoretical biology. The equations are given by

$$\left\{\begin{array}{rcl} \dot{x_1} &=& x_1(1-x_1) - f_1(x_1,x_2)\ , \\ \dot{x_2} &=& f_1(x_1,x_2) - d_1 x_2 - f_2(x_2,x_3)\ , \\ \dot{x_3} &=& f_2(x_2,x_3) - d_2 x_3\ , \end{array}\right. \tag{1}$$

with Holling Type-II functional responses

$$f_1(x_1,x_2) = \frac{a_1 x_1 x_2}{1+b_1 x_1}$$

and

$$f_2(x_2,x_3) = \frac{a_2 x_2 x_3}{1+b_2 x_2}\ .$$

This standard model has been studied by several authors, see *e.g.* Kuznetsov and Rinaldi (1996). The death rates d_1 and d_2 are often used as bifurcation parameters, with the other parameters set at $a_1 = 5$, $a_2 = 0.1$, $b_1 = 3$, and $b_2 = 2$. For these parameter values the model displays chaotic behaviour in a given parameter range of d_1 and d_2 [Hastings and Powell, 1991, Klebanoff and Hastings, 1994, McCann and Yodzis, 1995]. Before the onset of chaos, the system displays a series of period-doubling bifurcations. A starting point of this route to chaos can be found, for example, at $d_1 \approx 0.2080452$, $d_2 = 0.0125$, where there is a fold bifurcation in which two limit cycles appear.

Previous work by Boer et al. (1999, 2001) has shown that the parameter region where chaos occurs is intersected by homoclinic and heteroclinic global connections, and that this region is partly bounded by a homoclinic cycle-to-cycle connection. These results were obtained numerically using multiple shooting. In the previous paper (Doedel et al., 2007) we reproduced the results for the structurally stable heteroclinic point-to-cycle connection using the homotopy method. In this set-up we demonstrate how the homoclinic cycle-to-cycle connection can be detected, and continued in parameter space using the homotopy method. Also, it is set up such it can be used as well for a heteroclinic cycle-to-cycle connection.

2.1 Cycle data

The directory *01Cycle* provides the data for the cycle O in the cycle-to-cycle homoclinic connection. The command *make first* locates a Hopf bifurcation

```
BR    PT  TY LAB    PAR(1)
 1     1  EP   1  5.000000E-01 ...
 1     5  HB   2  5.122697E-01 ...
 1    10  EP   3  8.413455E-01 ...
```

Restarting from label 2, the command *make second* gives two uzer-defined points for $d_1 = 0.25$, $d_2 = 0.0125$.

```
BR    PT  TY LAB    PAR(1)            PERIOD
 2   108  PD   4  4.289115E-01 ... 6.588692E+01
 2   320  UZ   5  2.600001E-01 ... 5.080266E+01
 2   332  UZ   6  2.500004E-01 ... 4.895488E+01
 2   364  PD   7  2.166817E-01 ... 4.047737E+01
 2   389  LP   8  2.080452E-01 ... 3.437447E+01
 2   514  UZ   9  2.500000E-01 ... 2.428225E+01
 2   540  UZ  10  2.600000E-01 ... 2.340505E+01
 2   600  EP  11  2.840653E-01 ... 2.173427E+01
```

The data of the second uzer-defined point is exported.

2.2 Obtaining a starting point

To obtain a good starting point of the cycle-to-cycle connection orbit we need to know the direction of the eigenfunction. The command *make first* in the directory *02EigFunc* gives the logarithmic multipliers

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(13)
 1    50       2  2.428225E+01 ... -6.414969E+00  0.000000E+00
 1    51  BP   3  2.428225E+01 ... -6.414681E+00  0.000000E+00
 1   100       4  2.428225E+01 ... -1.514681E+00  0.000000E+00
 1   116  BP   5  2.428225E+01 ...  3.843325E-09  0.000000E+00
 1   121  BP   6  2.428225E+01 ...  4.399610E-01  0.000000E+00
 1   150       7  2.428225E+01 ...  3.339961E+00  0.000000E+00
 1   200  EP   8  2.428225E+01 ...  8.339962E+00  0.000000E+00
```

Restarting at label 6 with *make second*

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(13)
 2    50       9  2.428225E+01 ... 4.399610E-01  1.319472E-01
 2   100      10  2.428225E+01 ... 4.399610E-01  5.059062E-01
 2   150      11  2.428225E+01 ... 4.399610E-01  9.415602E-01
 2   157  UZ  12  2.428225E+01 ... 4.399610E-01  1.000000E+00
 2   200  EP  13  2.428225E+01 ... 4.399610E-01  1.394541E+00
```

We desire only the values of $v(0)$ of the uzer-defined point. The starting point is now calculated by taking the cycle coordinates at $t = 0$ ($(u_1, u_2, u_3) = (0.83978298543, 0.12528397115, 10.552876607)$) and making a small step in the direction of the unstable eigenfunction

$$u(0) = x(0) + \varepsilon v(0) \ , \tag{2}$$

where $i = 1, 2, 3$, $v(0)$ the coordinates of the unstable eigenfunction and $\varepsilon = -0.001$. The starting coordinates are then $(u_1, u_2, u_3) = (0.8397887445, 0.1252741872, 10.55323968)$.

3 Two adjoint eigenfunctions

Next we require the eigenfunctions of both cycles. The system is treated as if there are two separate cycles, though it is a homoclinic connection and the two cycles are one and the same. All formula's, boundary and integral conditions are copied, to a total of 12 ODE's, 14 boundary conditions and 2 integral conditions. According to the formula

$$n_{fp} = n_{bc} + n_{ic} + 1 - n_{fv} \tag{3}$$

this leaves 5 free parameters. Observe that there are six possible free parameters, and therefore in each run one of the multipliers is kept constant.

In the directory *03TwoAdjEigFunc* the command *make compute* generates the file *compute*. The command *@compute C6*, followed by the entry of an initial floquet multiplier value, for example, -0.9, will generate the starting file *s.rm3*.

The command *make first* results in three branching points for the "first" cycle

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(13)
 1   110  BP  12  2.428225E+01 ... -4.399607E-01  0.000000E+00 ...
 1   115  BP  13  2.428225E+01 ... -3.255862E-10  0.000000E+00 ...
 1   150      17  2.428225E+01 ...  3.500000E+00  0.000000E+00 ...
 1   180  BP  20  2.428225E+01 ...  6.414681E+00  0.000000E+00 ...
 1   200  EP  22  2.428225E+01 ...  8.414682E+00  0.000000E+00 ...
```

where $PAR(12)$ is the first Floquet multiplier, and the second floquet multiplier $PAR(14)$ is kept constant. The command *make second* then gives

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(13)
 2    10      23  2.428225E+01 ... -4.399610E-01  4.609158E-07 ...
 2    40      26  2.428225E+01 ... -4.399610E-01  2.747229E-01 ...
 2    62  UZ  29  2.428225E+01 ... -4.399610E-01  1.000000E+00 ...
 2    70      30  2.428225E+01 ... -4.399610E-01  1.363736E+00 ...
 2   100  EP  33  2.428225E+01 ... -4.399610E-01  3.060171E+00 ...
```

The command *make third* is started at the third branching point of the first run, which gives the unstable manifold, instead of the first branching point, which gives the stable manifold

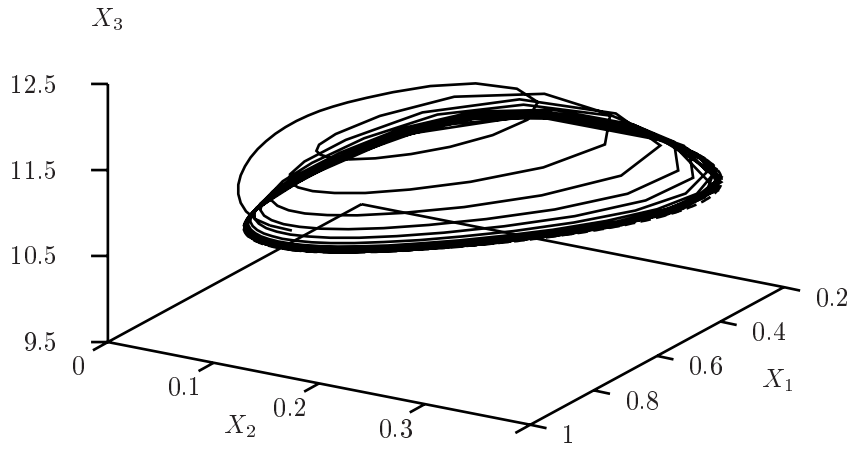

Figure 1: Plot of the approximate connection obtained using MATLAB. Observe the resolution is somewhat low.

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(14)
 3   110  BP  44  2.428225E+01 ... -4.399610E-01 -4.399606E-01 ...
 3   115  BP  45  2.428225E+01 ... -4.399610E-01 -3.254966E-10 ...
 3   150      49  2.428225E+01 ... -4.399610E-01  3.500000E+00 ...
 3   180  BP  52  2.428225E+01 ... -4.399610E-01  6.414681E+00 ...
 3   200  EP  54  2.428225E+01 ... -4.399610E-01  8.414682E+00 ...
```

The command *make fourth* then

```
BR    PT  TY LAB    PAR(11)           PAR(12)       PAR(14)
 4    30      57  2.428225E+01 ... -4.399610E-01  6.414681E+00 ...
 4    41  UZ  59  2.428225E+01 ... -4.399610E-01  6.414681E+00 ...
 4    60      61  2.428225E+01 ... -4.399610E-01  6.414681E+00 ...
 4   100  EP  65  2.428225E+01 ... -4.399610E-01  6.414681E+00 ...
```

The data of the last uzer-point is exported.

3.1 Obtaining a connection

From the directory *02EigFunc* we extract the starting coordinates

$$(u_1, u_2, u_3) = (0.8397887445, 0.1252741872, 10.55323968)\ .$$

The program CONTENT is then used to obtain the approximate connection time of the connecting orbit resulting from integration starting at this starting point. We use two base

points (it is a somewhat arbitrary choice which x_i-coordinate is used, but one should use two different ones for the two conditions). For the starting point we use a base point

$$\Phi[x^-] = x_1^- - 0.839783 \ ,$$

which is the x_1-coordinate, while for the end-point use a base-point

$$\Phi[x^+] = x_2^+ - 0.125284 \ ,$$

where the selected value is the starting x_2-coordinate of the limit cycle. A uzer-function has to be defined in CONTENT under *Select/Userfunctions*. Let the integration run until a suitable point is detected (most likely more detections will occur, but not all are relevant). The given time interval $T = 463.398$ is the initial period used in the following MATLAB integration.

The integration in MATLAB is done using two files, named *rm.m* and *process.m*, available in the directory *03TwoAdjEigFunc*. The first file contains the equations and parameter settings and the second file contains the starting data and integration routine. Observe that TC is the connection time obtained using CONTENT, while the used starting point is chosen such there is an approximate connection, shown in Figure 1.

3.2 Homotopy method

To improve the approximate connecting orbit the homotopy method is used in the directory *04Homotopy* (see Doedel, Friedman and Monteiro, 1994). The complete truncated BVP to be solved numerically consists of

$$\dot{x}^\pm - T^\pm f(x^\pm, \alpha) = 0 \ , \tag{4a}$$
$$x^\pm(0) - x^\pm(1) = 0 \ , \tag{4b}$$
$$\Phi[x^\pm] = 0 \ , \tag{4c}$$
$$\dot{w}^+ + T^+ f_u^{\mathrm{T}}(x^+, \alpha) w^+ + \lambda^+ w^+ = 0 \ , \tag{4d}$$
$$\dot{w}^- + T^- f_s^{\mathrm{T}}(x^-, \alpha) w^- + \lambda^- w^- = 0 \ , \tag{4e}$$
$$w^\pm(1) - s^\pm w^\pm(0) = 0 \ , \tag{4f}$$
$$\langle w^\pm(0), w^\pm(0) \rangle - 1 = 0 \ , \tag{4g}$$
$$\dot{u} - T f(u, \alpha) = 0, \tag{4h}$$
$$\langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle - h_{11} = 0 \ , \tag{4i}$$
$$\langle f(x^-(0), \alpha), u(0) - x^-(0) \rangle - h_{12} = 0 \ , \tag{4j}$$
$$\langle w^+(0), u(1) - x^+(0) \rangle - h_{21} = 0 \ , \tag{4k}$$
$$\langle w^-(0), u(0) - x^-(0) \rangle - h_{22} = 0 \ , \tag{4l}$$

where Φ in (4c) again defines any phase condition fixing the base points $x^\pm(0)$ on the cycles $O^\pm$, and h_{1k}, $k = 1, 2$, are *homotopy parameters*. We have 20 boundary conditions and 15 variables, thus there are six free continuation parameters: $\mu^\pm$, $T^\pm$, T (the time variable is

scaled to the unit interval $[0,1]$, so that both the cycle periods $T^{\pm}$ and the connecting time T become parameters), and, in turn, one of the four homotopy parameters $h_{11}, h_{12}, h_{21}, h_{22}$.

The continuation in AUTO07P can be done in four steps, where in each step one homotopy parameter is continued up to a zero. Several zeroes are detected for each homotopy parameters, where each zero correspond to a full rotation around the limit cycle. The command *make first* gives uzer-defined points for a continuation in (h_{11}, T)

BR	PT	TY	LAB	PAR(21)		PAR(7)	
1	40		3	-1.04335E-02	...	4.99335E+02	...
1	80		5	-1.10248E-02	...	4.99544E+02	...
1	120		7	-1.20982E-02	...	5.06367E+02	...
1	160		9	-8.85341E-03	...	5.18642E+02	...
1	188	UZ	11	-1.01205E-09	...	5.27176E+02	...
1	200		12	1.16970E-02	...	5.29915E+02	...
1	240		14	1.56837E-02	...	5.29375E+02	...
1	280		16	1.46523E-02	...	5.19410E+02	...
1	312	UZ	18	-1.23567E-13	...	5.08763E+02	...
1	320		19	-4.54740E-03	...	5.04691E+02	...
1	360		21	-1.06767E-02	...	4.99359E+02	...
1	400		23	-1.17754E-02	...	5.01232E+02	...
1	440		25	-1.11512E-02	...	5.13199E+02	...
1	480		27	-2.74047E-03	...	5.25539E+02	...
1	486	UZ	28	-2.95096E-14	...	5.27176E+02	...
1	500	EP	29	1.20778E-02	...	5.29910E+02	...

The command *make second*, restarting from label 8, results in uzer-defined points for a continuation in (h_{21}, T)

BR	PT	TY	LAB	PAR(23)		PAR(7)	
1	50		30	-1.05875E-02	...	5.24244E+02	...
1	52	UZ	31	1.22893E-11	...	5.23802E+02	...
1	100		32	5.29868E-02	...	5.05709E+02	...
1	144	UZ	33	-5.53957E-10	...	5.06528E+02	...
1	150		34	-5.56218E-02	...	5.07706E+02	...
1	200		35	-3.09866E-01	...	5.19771E+02	...
1	222	UZ	36	-1.01145E-09	...	5.25793E+02	...
1	242	UZ	37	1.11454E-09	...	5.30801E+02	...
1	250		38	-8.92395E-02	...	5.33284E+02	...
1	298	UZ	39	5.22099E-08	...	5.48106E+02	...
1	300	EP	40	4.22408E-02	...	5.49037E+02	...

Now, the homotopy is applied to the other side of the connecting orbit. The command *make third*, restarted at label 31, gives uzer-defined points for a continuation in (h_{12}, T)

```
 BR    PT  TY  LAB    PAR(22)          PAR(7)
  1    11  UZ   42   1.76914E-10 ... 5.06519E+02 ...
  1    40       45   1.73281E-03 ... 5.04644E+02 ...
  1    80       49   1.68136E-03 ... 5.02148E+02 ...
  1   105  UZ   52  -1.82353E-09 ... 5.02620E+02 ...
  1   120       54  -4.88450E-03 ... 5.04955E+02 ...
  1   160       58  -1.68828E-02 ... 5.13316E+02 ...
  1   200       62  -2.35839E-02 ... 5.22056E+02 ...
  1   240       66  -2.54848E-02 ... 5.30796E+02 ...
  1   300  EP   72  -2.64421E-02 ... 5.43906E+02 ...
```

Finally the command *make fourth*, restarted at label 42, yields several uzer-defined points for a continuation in (h_{22}, T)

```
 BR    PT  TY  LAB    PAR(24)          PAR(7)
 1    30       75  -2.03664E-02 ... 5.04151E+02 ...
 1    40  UZ   76   1.70343E-10 ... 5.02620E+02 ...
 1    60       78   1.78718E-01 ... 4.97136E+02 ...
 1    90       81   2.74721E-01 ... 4.88907E+02 ...
 1   115  UZ   84  -2.30068E-11 ... 4.82231E+02 ...
 1   120       85  -2.55786E-02 ... 4.80312E+02 ...
 1   128  UZ   86  -6.18816E-13 ... 4.78335E+02 ...
 1   150       89   3.11351E-01 ... 4.66268E+02 ...
 1   166  UZ   91   7.33349E-12 ... 4.57942E+02 ...
 1   178  UZ   93  -3.62980E-13 ... 4.54049E+02 ...
 1   180       94   1.85870E-02 ... 4.53051E+02 ...
 1   210       97   1.86843E-01 ... 4.38077E+02 ...
 1   219  UZ   98  -1.04706E-11 ... 4.33647E+02 ...
 1   231  UZ  101  -4.65764E-13 ... 4.29761E+02 ...
 1   240      102   1.39114E-01 ... 4.25269E+02 ...
 1   270      105   3.43323E-02 ... 4.10297E+02 ...
 1   272  UZ  106  -5.37645E-12 ... 4.09345E+02 ...
 1   284  UZ  108  -6.39627E-13 ... 4.05470E+02 ...
 1   300  EP  110   2.64699E-01 ... 3.97485E+02 ...
```

As can be seen, the connection time of the connecting orbit decreases, since the number of excursions near the cycle decreases. One of the connecting orbits where $h_{22} = 0$ is used in the continuation in α, and depicted in Figure 2, left panel. Observe that there are two types of orbits, examples of which are both depicted in Figure 2, right panel. One type, where the homotopy parameter is zero after first being positive, seems to be "in phase". The other type, where the homotopy parameter is zero after first being negative, is "out of phase". This latter type displays a "bump" at the starting section of the connection. Dispite this feature, this type produces the correct two-dimensional continuation curve of the limit point of the connecting orbit, as is demonstrated in the next subsection.

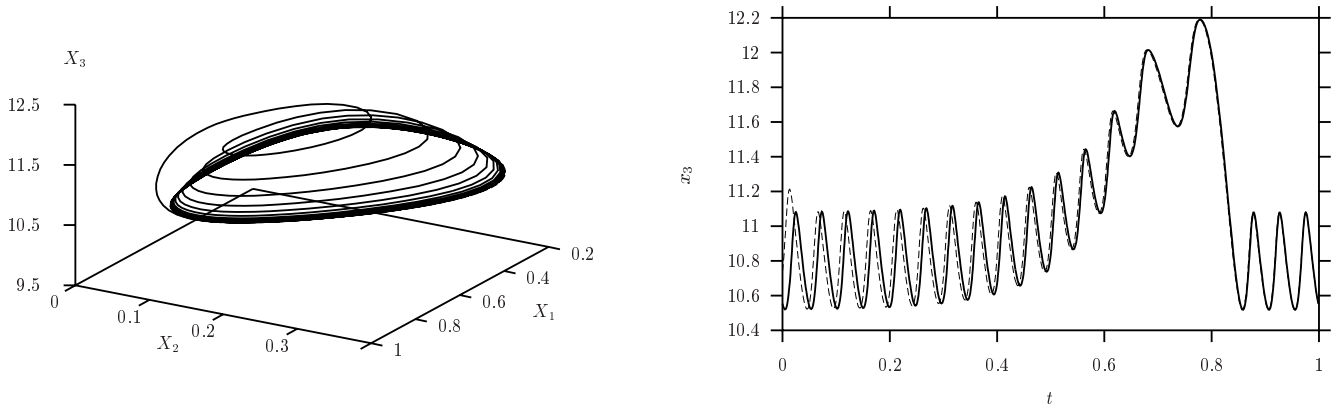

Figure 2: Left panel: Three-dimensional phase space plot of a cycle-to-cycle connecting orbit obtained with the homotopy method. Right panel: profiles (x_3 coordinate) of two connecting orbits obtained with the homotopy method, one "phased" (solid), and one "out of phase" (dashed). Observe the "bump" at the starting part of the latter orbit.

3.3 Continuation

The last directory *05Cont* demonstrates the detection and continuation of two branches of the fold bifurcation of the homoclinic cycle-to-cycle connection. There is a continuation of the connecting orbit in (α, T) up to a limit point, where α is one of the bifurcation parameters d_1 and d_2. Compared to the *fortran*-file in the directory *03Homotopy* we now have the slightly modified BVP

$$\dot{x}^{\pm} - T^{\pm} f(x^{\pm}, \alpha) = 0 \,, \tag{5a}$$
$$x^{\pm}(0) - x^{\pm}(1) = 0 \,, \tag{5b}$$
$$\dot{w}^{+} + T^{+} f_u^{\mathrm{T}}(x^{+}, \alpha) w^{+} + \lambda^{+} w^{+} = 0 \,, \tag{5c}$$
$$\dot{w}^{-} + T^{-} f_s^{\mathrm{T}}(x^{-}, \alpha) w^{-} + \lambda^{-} w^{-} = 0 \,, \tag{5d}$$
$$w^{\pm}(1) - s^{\pm} w^{\pm}(0) = 0 \,, \tag{5e}$$
$$\langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 = 0 \,, \tag{5f}$$
$$\dot{u} - T f(u, \alpha) = 0, \tag{5g}$$
$$\langle f(x^{+}(0), \alpha), u(1) - x^{+}(0) \rangle = 0 \,, \tag{5h}$$
$$\langle f(x^{-}(0), \alpha), u(0) - x^{-}(0) \rangle = 0 \,, \tag{5i}$$
$$\langle w^{+}(0), u(1) - x^{+}(0) \rangle = 0 \,, \tag{5j}$$
$$\langle w^{-}(0), u(0) - x^{-}(0) \rangle = 0 \,, \tag{5k}$$
$$||u(0) - x^{-}(0)||^2 - \varepsilon^2 = 0 \,, \tag{5l}$$

with one free parameter less. The parameter ε used in the calculation of the starting point of the approximate connection is fixed in the below continuations. We can now do a continuation in α_1 and subsequently in (α_1, α_2) for fixed T after the detection of a limit

point, using the standard fold-detection facilities of AUTO.

In the first run *make first* we improve our approximate connecting orbit, obtained from label 108 of the fourth run in the homotopy directory, by increasing the connection period T

```
 BR    PT  TY  LAB    PAR(7)
  1    30        4   4.08132E+02 ...
  1    60        7   4.17069E+02 ...
  1    90       10   4.25010E+02 ...
  1   120       13   4.30414E+02 ...
  1   150       16   4.36979E+02 ...
  1   180       19   4.51841E+02 ...
  1   210       22   4.55936E+02 ...
  1   240       25   4.68747E+02 ...
  1   270       28   4.78542E+02 ...
  1   300  EP   31   4.83855E+02 ...
```

Now we start the continuation in d_2 with fixed period *make second* from label 20 and limit points are found

```
 BR    PT  TY  LAB    PAR(1)           PAR(3)
  1   165  UZ   32   2.78500E-01 ... 2.20782E+01 ...
  1   185  LP   33   2.80908E-01 ... 2.19267E+01 ...
  1   241  UZ   34   2.78500E-01 ... 2.20782E+01 ...
  1   262  LP   36   2.77691E-01 ... 2.21300E+01 ...
  1   273  UZ   37   2.78500E-01 ... 2.20782E+01 ...
  1   283  LP   38   2.80913E-01 ... 2.19265E+01 ...
  1   294  UZ   39   2.78500E-01 ... 2.20782E+01 ...
  1   512  LP   40   2.30598E-01 ... 2.65515E+01 ...
  1  1000  EP   49   2.70309E-01 ... 2.26269E+01 ...
```

Restarting at the limit points in will yield two homoclinic cycle-to-cycle bifurcation curves in (d_1, d_2). First, however, the 2D continuation must be prepared from label 40 by *make third*

```
 BR    PT  TY  LAB    PAR(1)           PAR(2)
  2     5  EP   44   2.30598E-01 ... 1.25000E-02 ...
```

We can now start with the continuation in 2D. With *make fourth* the primary homoclinic branch is detected

```
 BR    PT  TY  LAB    PAR(1)           PAR(2)
  2   100       51   2.51318E-01 ... 1.07436E-02 ...
  2   200       52   2.59487E-01 ... 1.03773E-02 ...
  2   300       53   2.68031E-01 ... 1.02636E-02 ...
```

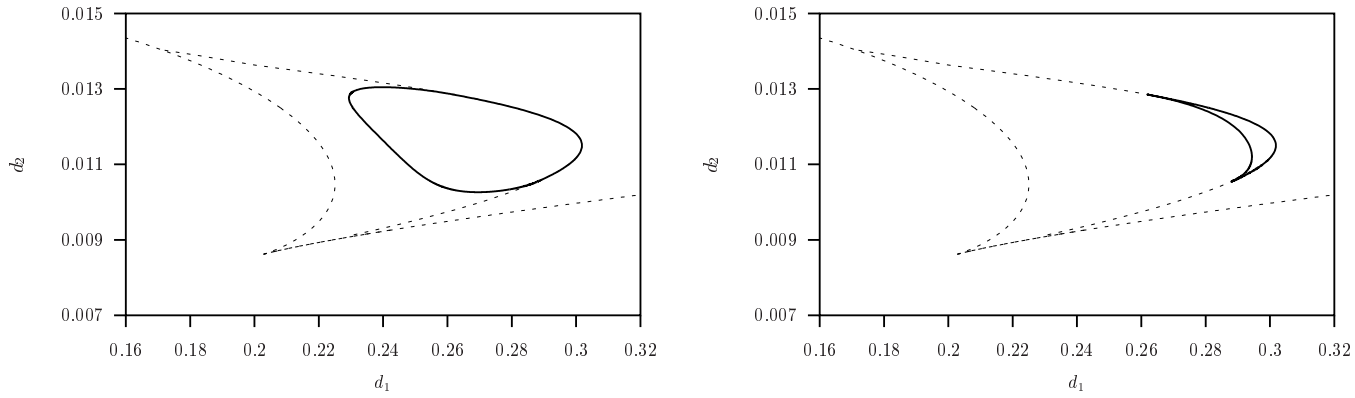

Figure 3: Two-parameter curves of the primary (left) and secondary (right) homoclinic cycle-to-cycle connections of the food chain model.

```
 2   400        54   2.77963E-01 ... 1.03168E-02 ...
 2   500        55   2.84465E-01 ... 1.04346E-02 ...
 2   592  MX    56   2.87786E-01 ... 1.05339E-02 ...
```

With *make fifth* we restart at label 33 for the preparation of the 2D continuation of the secondary branch

```
BR    PT  TY  LAB    PAR(1)           PAR(2)
 2     5  EP   50   2.80908E-01 ... 1.25000E-02 ...
```

With *make sixth* we obtain part of the secondary branch

```
BR    PT  TY  LAB    PAR(1)           PAR(2)
 2   100        51   2.62585E-01 ... 1.28404E-02 ...
 2   116  LP    52   2.61790E-01 ... 1.28531E-02 ...
 2   200        53   2.92202E-01 ... 1.17084E-02 ...
 2   232  LP    54   2.94383E-01 ... 1.12019E-02 ...
 2   300        55   2.92705E-01 ... 1.08177E-02 ...
 2   400        56   2.88873E-01 ... 1.05765E-02 ...
 2   427  MX    57   2.88193E-01 ... 1.05491E-02 ...
```

The remainders of the two branches are found by *make seventh*

```
BR    PT  TY  LAB    PAR(1)           PAR(2)
 2    38  LP   51   2.29368E-01 ... 1.27579E-02 ...
 2   100        52   2.32914E-01 ... 1.29992E-02 ...
 2   200        53   2.57779E-01 ... 1.29115E-02 ...
 2   300        54   2.99234E-01 ... 1.18721E-02 ...
 2   319  LP    55   3.01847E-01 ... 1.15004E-02 ...
 2   400        56   2.90814E-01 ... 1.06520E-02 ...
 2   493  MX    57   2.87940E-01 ... 1.05396E-02 ...
```

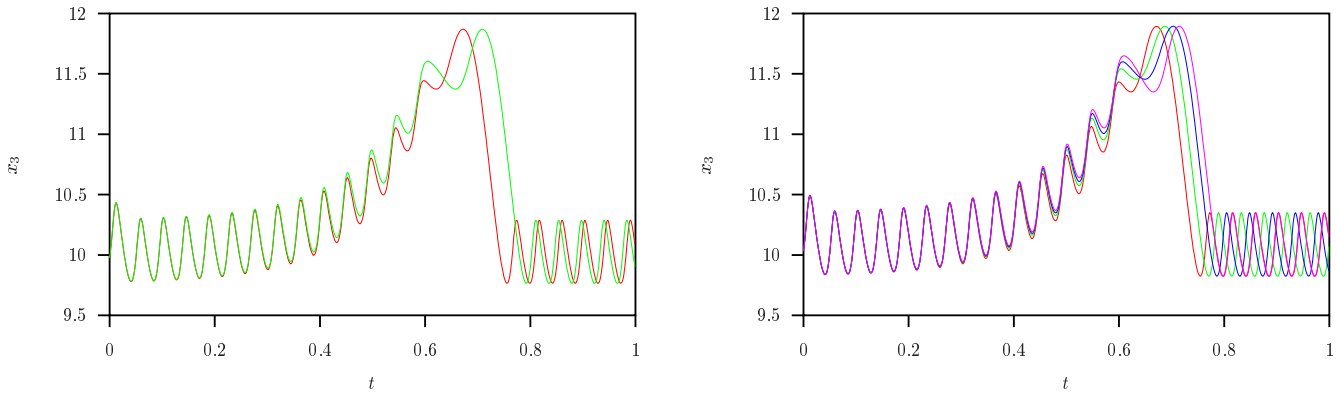

Figure 4: Profile plots of the four connecting homoclinic orbits that exist at $d_1 = 0.2785, d_2 = 0.0125$.

and *make eighth*

BR	PT	TY	LAB	PAR(1)		PAR(2)	
2	65	LP	51	3.01846E-01	...	1.15003E-02	...
2	100		52	3.00208E-01	...	1.11876E-02	...
2	200		53	2.95315E-01	...	1.08583E-02	...
2	215	LP	54	2.94700E-01	...	1.08270E-02	...
2	300		55	2.90388E-01	...	1.06346E-02	...
2	400		56	2.88242E-01	...	1.05509E-02	...
2	445	LP	57	2.87983E-01	...	1.05412E-02	...
2	454	MX	58	2.88048E-01	...	1.05436E-02	...

The two-dimensional continuation curve of the primary homoclinic cycle-to-cycle connection is depicted in Figure 3, left panel. The two-dimensional continuation curve of the secondary branch is depicted in the right panel of the same Figure. The two branches partly overlap each other. That they are indeed two different branches follows from the profiles of the four different connecting orbits that exist at the same value of $d_1 = 0.2785, d_2 = 0.0125$, see Figure 4, right panel, while there are two distinct connecting orbits at the same limit point ($d_1 = 0.2809078, d_2 = 0.0125$, left panel).

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