Detection and continuation of a heteroclinic point–to–cycle connection in the Lorenz equations.

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Abstract

Accompanying manuscript to the demonstration of the detection and continuation of a heteroclinic point-to-cycle connection in the 3D Lorenz model with the bifurcation software package AUTO, by use of the homotopy method described in the paper. The files are downloadable from
`http://www.bio.vu.nl/thb/research/project/globif`.

1 Disclaimer

The following results have been obtained under Sun Solaris 8, using a FORTRAN *compiler f77 for* AUTO97*, and using a* FORTRAN *compiler f95 for* AUTO07P*. The results might differ slighthy using a different compiler or a different version of* AUTO*.*

2 Lorenz equations

Consider the well-known three-dimensional Lorenz system

$$\begin{cases} \dot{x} &= \sigma(y-x) , \\ \dot{y} &= rx - y - xz , \\ \dot{z} &= xy - bz , \end{cases} \tag{1}$$

where σ is usually 10, b often is set to 8/3 and r is the principal bifurcation parameter. At $r = 1$ there is a supercritical pitchfork bifurcation that gives rise to two symmetric nontrivial equilibria. Two symmetric saddle cycles appear (together with a nontrivial hyperbolic invariant set) from the primary homoclinic orbits to the origin at $r = 13.962\ldots$. The Hopf bifurcation at $r = 24.7368\ldots$ is a subcritical Hopf bifurcation of nontrivial equilibria.

At a critical value r_{het} there is a heteroclinic point-to-cycle connection, that generates a chaotic attractor. Its domain of attraction is bounded by the stable invariant manifolds of the saddle cycles. Beyn (1990) approximated r_{het} to be 24.05, later Dieci and Rebaza (2004) calculated

$$r_{het} = 24.057900322267.$$

The heteroclinic connection can be continuated in two parameters, for example r and σ, with b fixed. The corresponding curve in the (r, σ)-plane was first shown in Appendix II written by L.P. Shil'nikov to the Russian translation of the book by Marsden and McCraken (see Pampel (2001), Dieci and Rebaza, 2004, for more recent results on the two-parameter continuation). Here we demonstrate the use of the homotopy method in AUTO97 for locating and continuation of this heteroclinic connection.

2.1 Cycle

The directory *01Cycle* gives the starting file *s.C4*, obtained after manual extraction following *make first*

```
BR    PT  TY LAB    PAR(1)           PERIOD
 1     5        2  2.100735E+01 ... 8.158130E-01
 1    10        3  2.100236E+01 ... 8.160907E-01
 1    11  UZ    4  2.100000E+01 ... 8.162224E-01
 1    15        5  2.091620E+01 ... 8.209208E-01
 1    20        6  2.021871E+01 ... 8.625092E-01
```

1	25		7	1.838317E+01	...	9.999038E-01
1	30	EP	8	1.557878E+01	...	1.390823E+00

The data file gives a saddle limit cycle O^+ of (1) at $r = 21$, with the base point

$$x^+(0) = (9.265335, 13.196014, 15.997250)$$

and period $T^+ = 0.816222$.

2.2 Eigendata

To obtain the eigenfunction of the cycle the file *s.C4* is transported to directory *02AdjEigenFunc*. The command *make compute* is required to generate a program for creating a starting file. The command *@compute C4* then generates a starting file. A starting Floquet multiplier must be entered, say, 1.1, and the file *s.fl* is generated.

The command *make first* continues the trivial solution of the appropriate BVP (see the paper), until a branch point with

$$\lambda = \ln(\mu_u^+) = 0.231854$$

is located (label 8)

BR	PT	TY	LAB	PAR(11)		PAR(12)	PAR(13)
1	50		5	8.162224E-01	...	6.468125E-01	0.000000E+00
1	100		6	8.162224E-01	...	1.468124E-01	0.000000E+00
1	115	BP	7	8.162224E-01	...	6.997552E-09	0.000000E+00
1	139	BP	8	8.162224E-01	...	-2.318538E-01	0.000000E+00
1	150		9	8.162224E-01	...	-3.418544E-01	0.000000E+00
1	200	EP	10	8.162224E-01	...	-8.418545E-01	0.000000E+00

From this point, a nontrivial branch can be continued with the command *make second*

BR	PT	TY	LAB	PAR(11)		PAR(12)	PAR(13)
2	10		11	8.162224E-01	...	-2.318538E-01	5.061320E-03
2	20		12	8.162224E-01	...	-2.318538E-01	1.446626E-01
2	30		13	8.162224E-01	...	-2.318538E-01	8.280846E-01
2	32	UZ	14	8.162224E-01	...	-2.318538E-01	1.000000E+00
2	40		15	8.162224E-01	...	-2.318538E-01	4.338433E+00
2	50	EP	16	8.162224E-01	...	-2.318538E-01	1.465519E+01

The value $h = 1$ can be reached, which gives a nontrivial eigenfunction with $\|w(0)\| = 1$, namely

$$w(0) = (0.168148, 0.877764, -0.448616).$$

In these continuations, all parameters r, σ, and b are fixed. The data from the q-file is exported as *s.CE30*.

2.3 Approximate connecting orbit

The reduced file *CE.dat* is required to obtain an approximation of the heteroclinic connection with the use of MATLAB. For this, we consider another BVP with

$$\Psi[x^+] = x_1^+(0) - 9.265335 ,$$

also described in the paper, and continue its solution at fixed system parameters with respect to (T, h_1). Figure 1 shows three consecutive solutions with $h_1 = 0$. The end point of the last solution (with $T = 2.00352$) is located near the base point $x^+(0)$ of the cycle O^+.

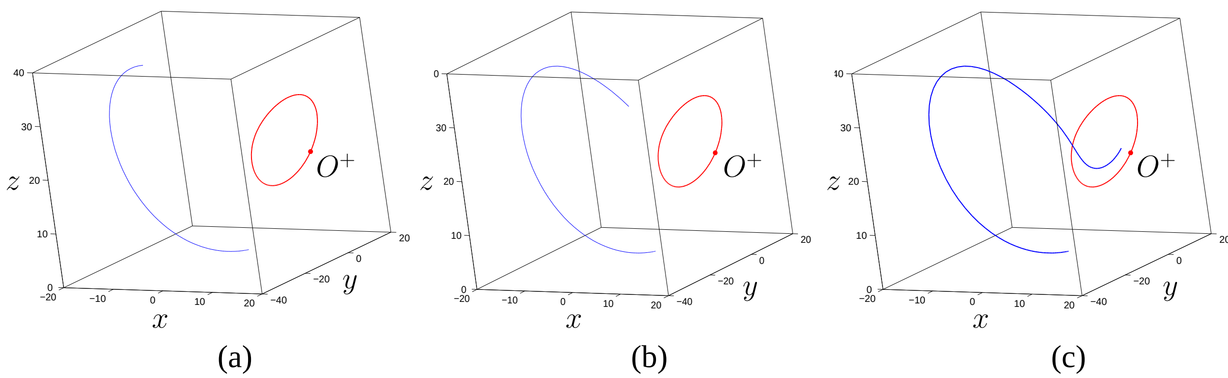

Figure 1: Continuation in T: (a) $T = 1.43924$; (b) $T = 1.54543$; (c) $T = 2.00352$.

2.4 First homotopy step

Using the obtained solution we can detect and continue a heteroclinic point-to-cycle connection. For that we first improve the approximate connection by using the homotopy method (see paper).

The first step in improving the approximate connection is demonstrated in the directory *03HomotopyH1*. The connection is improved with regard to (h_1, T) with the command *make first*, which results in

```
 BR    PT  TY LAB    PAR(21)          PAR(13)
  1    30        2 -2.077886E+03 ... 6.556384E-01 ...
  1    60        3 -2.134607E+03 ... 9.797939E-01 ...
  1    90        4 -2.431542E+03 ... 1.233973E+00 ...
  1   120        5 -2.426424E+03 ... 1.244395E+00 ...
  1   150        6 -2.335221E+03 ... 1.278126E+00 ...
  1   180        7 -8.836567E+02 ... 1.380214E+00 ...
  1   189  UZ    8 -1.448613E-10 ... 1.439241E+00 ...
```

```
  1   210       9  1.438202E+02 ... 1.484886E+00 ...
  1   240      10  1.168608E+02 ... 1.504343E+00 ...
  1   258  UZ  11 -2.346855E-10 ... 1.545430E+00 ...
  1   270      12 -5.459888E+02 ... 1.779221E+00 ...
  1   300      13 -5.315282E+02 ... 1.850416E+00 ...
  1   330      14 -1.103569E+02 ... 1.983043E+00 ...
  1   333  UZ  15 -6.856820E-07 ... 2.003518E+00 ...
  1   360  EP  16  5.854517E+02 ... 2.155057E+00 ...
```

where the uzer-defined points are $h_1 = 0$. We select the third labelled point.

2.5 Second homotopy step

The second homotopy step is demonstrated in the directory *04HomotopyH2*. The starting file is taken from the data of the third uzer-defined point of the run from the previous directory. The command *make first* yields

```
 BR    PT  TY LAB    PAR(1)          PAR(11)           PAR(12)
  1     5       2  2.162229E+01 ... 7.831003E-01 ... -1.780327E-01 ...
  1    10       3  2.250246E+01 ... 7.409380E-01 ... -1.143299E-01 ...
  1    15       4  2.335425E+01 ... 7.045150E-01 ... -6.386583E-02 ...
  1    20  UZ   5  2.405790E+01 ... 6.771718E-01 ... -2.891096E-02 ...
  1    25       6  2.466939E+01 ... 6.551659E-01 ... -2.681725E-03 ...
  1    30       7  2.473231E+01 ... 6.529867E-01 ... -1.789118E-04 ...
  1    35       8  2.456283E+01 ... 6.588915E-01 ... -7.000313E-03 ...
  1    40  EP   9  2.400646E+01 ... 6.790946E-01 ... -3.128437E-02 ...
```

where the uzerpoint is $PAR(21) = h_2 = 0$. The data from this point is a good approximation of the heteroclinic point-to-cycle connection to start the continuation from.

2.6 Continuation

Before the continuation of the connection is started we first improved the connection with the use of a new BVP. The length of the connecting orbit is increased by the continuation in (r, T) with the command *make first*

```
 BR    PT  TY LAB    PAR(13)         PAR(1)
  1    10       2  2.180071E+00 ... 2.405790E+01 ...
  1    20       3  2.460715E+00 ... 2.405790E+01 ...
  1    30       4  2.780010E+00 ... 2.405790E+01 ...
  1    38  UZ   5  3.000000E+00 ... 2.405790E+01 ...
  1    40       6  3.096544E+00 ... 2.405790E+01 ...
  1    50       7  3.333010E+00 ... 2.405790E+01 ...
  1    60       8  3.641021E+00 ... 2.405790E+01 ...
```

1	70		9	3.868541E+00	...	2.405790E+01	...
1	75	UZ	10	4.000000E+00	...	2.405790E+01	...
1	80		11	4.143823E+00	...	2.405790E+01	...
1	90		12	4.434139E+00	...	2.405790E+01	...
1	100		13	4.677092E+00	...	2.405789E+01	...
1	110		14	4.989955E+00	...	2.405789E+01	...
1	111	UZ	15	5.000000E+00	...	2.405789E+01	...
1	120		16	5.269551E+00	...	2.405786E+01	...
1	130		17	5.565366E+00	...	2.405787E+01	...
1	140		18	5.832456E+00	...	2.405773E+01	...
1	150	EP	19	6.092645E+00	...	2.405781E+01	...

The user-point for $T = 3.0$ (label 5) gives the profile as shown in Figure 3(b), which is a good approximation for r_{het}.

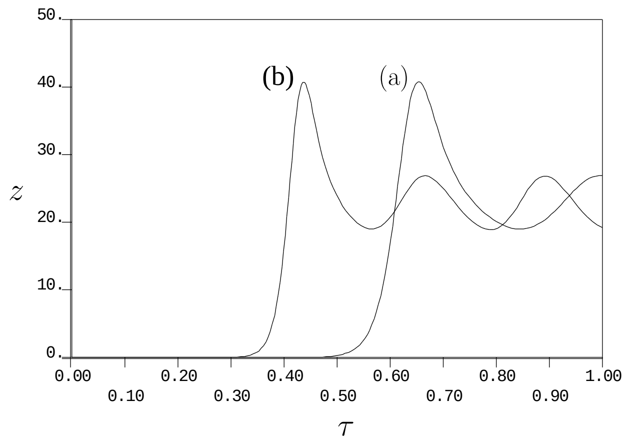

Figure 2: Two profiles of the truncated connecting orbit in the Lorenz system scaled to the unit time interval: (a) $T = 2.00352$; (b) $T = 3.0$.

With the final steps we do a continuation in two system parameters (r, σ) with T. Running *make second* when restarting at label 5 gives the forward continuation

BR	PT	TY	LAB	PAR(1)		PAR(3)	
1	60		20	2.532580E+01	...	1.315854E+01	...
1	120		21	2.848590E+01	...	1.937285E+01	...
1	180		22	3.213813E+01	...	2.736677E+01	...
1	240		23	3.679986E+01	...	3.899747E+01	...
1	300		24	4.226648E+01	...	5.388082E+01	...
1	360		25	4.810053E+01	...	7.068921E+01	...
1	420		26	5.423869E+01	...	8.921075E+01	...
1	450	EP	27	5.788374E+01	...	1.006821E+02	...

The command *make third* gives the backward continuation

BR	PT	TY	LAB	PAR(1)		PAR(3)	
1	60		20	2.536384E+01	...	6.714805E+00	...
1	120		21	3.522998E+01	...	4.722289E+00	...
1	180		22	4.194898E+01	...	4.279005E+00	...
1	240		23	4.785003E+01	...	4.024580E+00	...
1	300		24	5.325330E+01	...	3.849727E+00	...
1	360		25	5.837942E+01	...	3.716588E+00	...
1	420		26	6.333877E+01	...	3.609165E+00	...
1	480		27	6.818296E+01	...	3.519435E+00	...
1	540		28	7.288567E+01	...	3.443541E+00	...
1	600		29	7.751923E+01	...	3.377460E+00	...
1	660		30	8.209401E+01	...	3.319200E+00	...
1	720		31	8.661038E+01	...	3.267393E+00	...
1	780		32	9.107042E+01	...	3.220976E+00	...
1	840		33	9.547054E+01	...	3.179166E+00	...
1	900		34	9.980344E+01	...	3.141366E+00	...
1	960	EP	35	1.040616E+02	...	3.107088E+00	...

The resulting two-parameter curve depicted in Figure 3 corresponds to the point-to-cycle connection in (1).

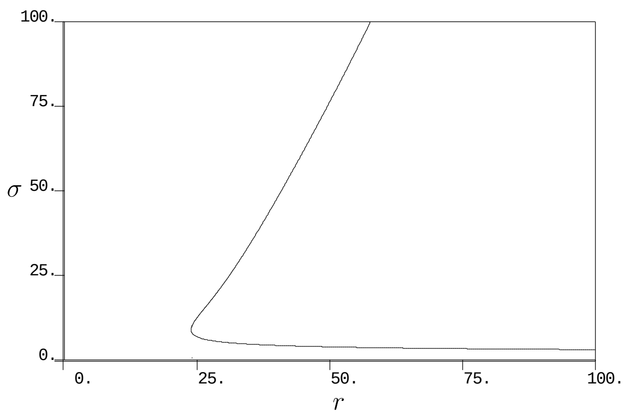

Figure 3: The bifurcation curve of the Lorenz system corresponding to the point-to-cycle connection.

3 Acknowledgements

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