Detection and continuation of a heteroclinic point–to–cycle connection in a food chain model.

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Abstract

Accompanying manuscript to the demonstration of the detection and continuation of a heteroclinic point-to-cycle connection in the 3D food chain model by Rosenzweig–MacArthur with the bifurcation software package AUTO, by use of the homotopy method described in the paper. The files are downloadable from http://www.bio.vu.nl/thb/research/project/globif.

1 Disclaimer

The following results have been obtained under Sun Solaris 8, using a FORTRAN *compiler f77 for* AUTO97*, and using a* FORTRAN *compiler f95 for* AUTO07P*. The results might differ slighthy using a different compiler or a different version of* AUTO*.*

2 Food chain model

A standard three-level food chain model from the field of theoretical biology is the scaled Rosenzweig-MacArthur system (1963)

$$\left\{ \begin{array}{rcl} \dot{x} & = & x(1-x) - f_1(x,y) \ , \\ \dot{y} & = & f_1(x,y) - d_1 y - f_2(x,y) \ , \\ \dot{z} & = & f_2(x,y) - d_2 z \ , \end{array} \right. \tag{1}$$

with Holling Type-II functional responses

$$f_1(x,y) = \frac{a_1 xy}{1+b_1 x} \ , \ f_2(x,y) = \frac{a_2 yz}{1+b_2 y} \ .$$

The model has already been analyzed extensively by many authors, see for instance Kuznetsov and Rinaldi (1996). The default parameter values are

$$a_1 = 5, a_2 = 0.1, b_1 = 3, b_2 = 2,$$

with d_1 and d_2 as bifurcation parameters. Boer et al. (2001) showed that at least two structurally stable heteroclinic point-to-cycle connections can be found in this system, that exist in a region of two-parameter space. This region is bounded at one side by a tangent bifurcation where the two heteroclinic connections merge. At the other side it is bounded by a tangent bifurcation of the limit cycle involved in the heteroclinic connection. Figure 1 displays a two-parameter bifurcation diagram of the food chain model in which these two tangent bifurcations are depicted.

In this AUTO demo we show that we can approximate a heteroclinic point-to-cycle connection by using the homotopy method described in the paper.

2.1 Local bifurcation analysis

The directory *01Cycle* contains the material to obtain the relevant cycle for the point-to-cycle connection. The starting point and starting parameter values are

$$d_1 = 0.5, d_2 = 0.0125, x_1 = 0.741582, x_2 = 0.166667, x_3 = 8.664399.$$

Note we use the data of the point with period $T = 24.28225$.

The command *make first* gives a uzer-defined equilibrium at $d_1 = 0.25$ for a backward one-dimensional continuation in d_1

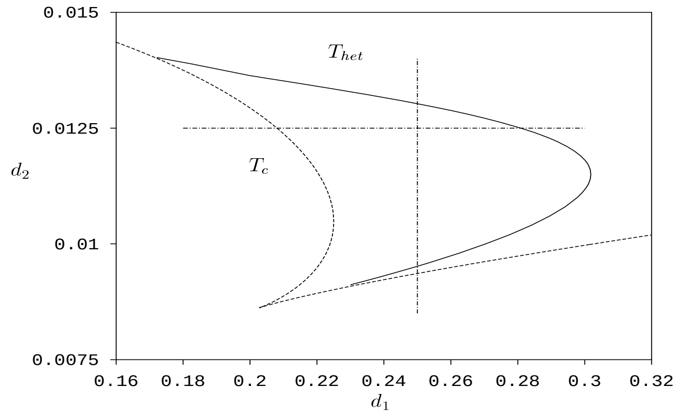

Figure 1: Two-parameter bifurcation diagram of the food chain model that indicates the region where there exist point-to-cycle connections. The region is bounded on one side by the tangent for the cycle (T_c) and on the other side by the curve T_{het}.

```
  BR    PT  TY  LAB    PAR(1)
   1     1  EP    1   5.00000E-01 ...
   1     9  UZ    2   2.50000E-01 ...
   1    10  EP    3   1.00420E-01 ...
```

This point, $\xi = (0.74158162, 0.166667, 11.997732)$, will be required in the continuation of the connecting orbit. A forward one-dimensional continuation in d_1 with *make second* gives a Hopf bifurcation

```
  BR    PT  TY LAB    PAR(1)
   1     1  EP   1  5.000000E-01 ...
   1     5  HB   2  5.122697E-01 ...
   1    10  EP   3  8.413455E-01 ...
```

from which we restart *make third*

```
  BR    PT  TY LAB    PAR(1)            PERIOD
   2   108  PD   4  4.289115E-01 ... 6.588692E+01
   2   331  UZ   5  2.500004E-01 ... 4.895489E+01
   2   363  PD   6  2.166817E-01 ... 4.047737E+01
   2   388  LP   7  2.080452E-01 ... 3.437447E+01
   2   513  UZ   8  2.500000E-01 ... 2.428225E+01
   2  1000  EP   9  4.310418E-01 ... 1.659429E+01
```

The data of the second uzer-defined point is exported.

2.2 Adjoint eigenfunction

In the directory *02AdjEigFunc* the logarithmic adjoint eigenfunction is calculated. For this, first the file *compute** must be created by typing *make compute*. Then, after the command

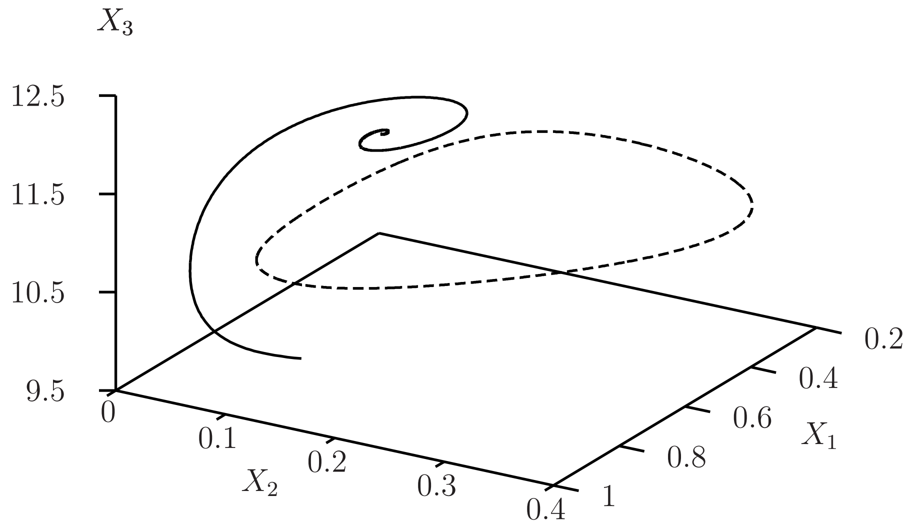

Figure 2: Approximation of a heteroclinic point-to-cycle connection in the Rosenzweig–MacArthur model, after time-integration in MATLAB.

@compute C6 the uzer is asked to enter a Floquet multiplier, for instance 1.1. With this command the start-up file is expanded with a starting Floquet eigenvalue and zeroes for the eigenfunction.

The output is generated by typing *make first*, that results in two branching points

```
BR    PT  TY LAB    PAR(11)             PAR(12)       PAR(13)
 1   121  BP   2  2.428225E+01 ... -5.143422E-06  0.000000E+00
 1   165  BP   3  2.428225E+01 ... -4.399591E-01  0.000000E+00
 1   200  EP   4  2.428225E+01 ... -7.899592E-01  0.000000E+00
```

where $PAR(12)$ is the log multiplier. The first BP obviously is zero, so we restart at the second BP label 3 with *make second*, which results in

```
BR    PT  TY LAB    PAR(11)             PAR(12)       PAR(13)
 2   452  UZ   5  2.428225E+01 ... -4.399611E-01  1.000000E+00
 2   500  EP   6  2.428225E+01 ... -4.399611E-01  1.212898E+00
```

The uzer-point given $PAR(13) = 1$, which gives the eigenfunction for the attracting multiplier. The file *out.dat* is created from the uzer-point data in the q-file.

2.3 Calculating the connection

To obtain a sufficient initial approximate connecting orbit we first need a starting point. The starting point is calculated by splitting the adjoint stable vector (evaluated at $d_1 = 0.25, d_2 = 0.0125$)

$$v = (9.844010 \cdot 10^{-2}, 0.168771, 4.953227 \cdot 10^{-3})^T$$

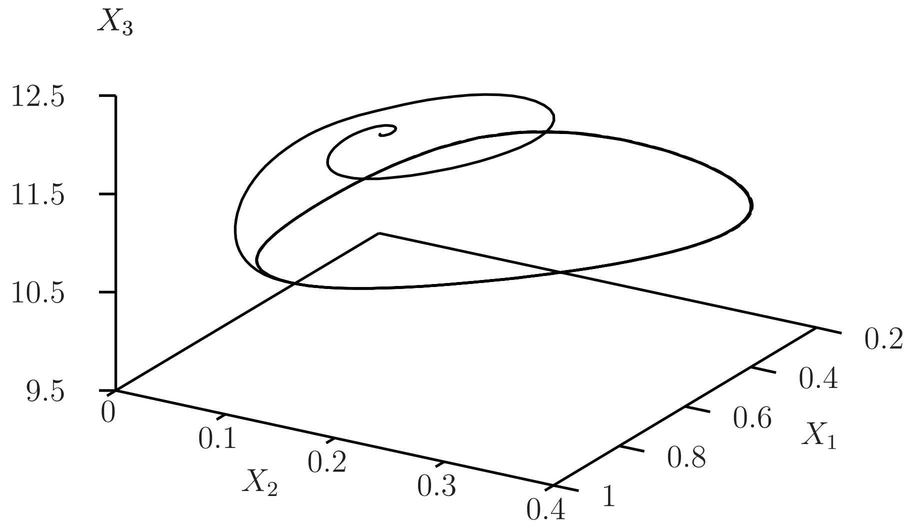

Figure 3: A heteroclinic point-to-cycle connection in the Rosenzweig–MacArthur model, obtained with the homotopy method.

into $v^{(1)}$ and $v^{(2)}$, as described in the paper.After normalization, and multiplying by a small ε, say 0.001, the starting point is

$$x_1 = 0.7424451445, x_2 = 0.1661629904, x_3 = 11.99773243.$$

A base point is determined using CONTENT, and using integration in MATLAB with the cycle period $T_C = 155.905$ results in the file *s.rm3*. This file is included in the directory. The connection now looks like in Figure 2.

2.4 First homotopy step

The directory *03HomotopyH1* contains the first homotopy step, as explained in the paper. In the first homotopy step a BVP combination of Eqn. (3,4a,5) is solved (see Appendix). With *make first* it is attempted to find a point where the first homotopy parameter equals zero by increasing the period

```
 BR    PT  TY LAB    PAR(16)          PAR(13)
  1    10       2  2.059015E-02 ... 1.564619E+02 ...
  1    20       3  5.486775E-02 ... 1.624831E+02 ...
  1    25  UZ   4  9.999968E-03 ... 1.674593E+02 ...
  1    30       5  7.888757E-03 ... 1.724544E+02 ...
  1    31  UZ   6  1.000000E-02 ... 1.733970E+02 ...
  1    40       7  4.301262E-02 ... 1.823879E+02 ...
  1    50  EP   8  1.281478E-02 ... 1.923758E+02 ...
```

but such a point is not found yet. Instead, the data at label 5 is used as a starting point for the next homotopy step.

2.5 Second homotopy step

A different BVP, a combination of Eqn. (3,4,6), is used in the second homotopy step to obtain $h_1 = 0$ (see the Appendix and the paper) in the directory *04HomotopyH2*. A continuation is done in $(c1, c2, h1)$ with *make first*

```
BR    PT  TY  LAB    PAR(16)          PAR(14)      PAR(15)
 1    10        2   7.02548E-03 ...  9.62015E-01  2.72998E-01 ...
 1    17  UZ    3  -8.67130E-06 ...  7.54119E-01  6.56738E-01 ...
 1    20        4  -2.74817E-02 ...  3.71497E-01  9.28434E-01 ...
 1    30  EP    5  -1.41536E-02 ... -8.62066E-01 -5.06796E-01 ...
```

with a solution at label 3. This data is exported.

2.6 Third homotopy step

In the directory *05HomotopyH3* a continuation is done in $(c1, c2, h2)$ with *make first*

```
BR    PT  TY  LAB    PAR(16)         PAR(14)      PAR(15)
 1    10        2   1.41735E-01 ... 7.40483E-01  6.72075E-01 ...
 1    20        3   6.92385E-02 ... 6.94232E-01  7.19751E-01 ...
 1    30        4   3.34368E-03 ... 6.26538E-01  7.79391E-01 ...
 1    31  UZ    5  -1.28187E-07 ... 6.21060E-01  7.83763E-01 ...
 1    40  EP    6  -2.56575E-02 ... 5.55436E-01  8.31559E-01 ...
```

where a uzer-defined point is found where $h_2 = 0$.

Continuation in one or more system parameters starting from a file extracted from this uzer-point is subject to the detection of spurious limit points. We therefore replace the BC

$$\langle v, u(0) - \xi \rangle = 0 \, ,$$
$$\langle \eta, u(0) - \xi \rangle - \kappa = 0 \, ,$$

with a new BC (see paper and next subsection) which requires a fourth homotopy step.

2.7 Fourth homotopy step

A final homotopy step is required to find $\kappa = 0$ for the newly introduced BC

$$\big(u_2(0) - \xi_2\big) - \kappa \, ,$$

that describes an intersection plane through the equilibrium ξ for $\kappa = 0$. In the directory *06HomotopyH4* the command *make first* finds such a point where the distance of the connection to the x_2-coordinate of the equilibrium is zero

```
BR    PT  TY  LAB    PAR(32)          PAR(11)
  1     5         2  -2.77771E-04 ... 2.42822E+01 ...
  1    10         3  -2.45286E-04 ... 2.42822E+01 ...
  1    15         4  -2.02452E-04 ... 2.42822E+01 ...
  1    20         5  -1.52859E-04 ... 2.42822E+01 ...
  1    25         6  -1.19046E-04 ... 2.42822E+01 ...
  1    30         7  -8.71254E-05 ... 2.42822E+01 ...
  1    35         8  -4.76595E-05 ... 2.42822E+01 ...
  1    40         9  -1.29396E-05 ... 2.42822E+01 ...
  1    43  UZ    10   1.43015E-13 ... 2.42822E+01 ...
  1    45        11   1.03321E-05 ... 2.42822E+01 ...
  1    50  EP    12   2.64997E-05 ... 2.42822E+01 ...
```

gives a uzer-defined point for $\kappa = 0$. We now have a good approximation of the point-to-cycle connecting orbit. This connection is depicted in Figure 3.

2.8 Continuation

The continuation of the point-to-cycle connection has only been successfully tested under AUTO07P. It requires a new BVP, described in the paper. The demo in the directory *07Cont* shows the following results.

First, the point-to-cycle connection time is increased with *make first*

```
BR    PT  TY  LAB    PAR(1)          PAR(11)
  1    50         2   1.92056E+02 ... 2.42822E+01 ...
  1   100         3   2.13635E+02 ... 2.42822E+01 ...
  1   150         4   2.32940E+02 ... 2.42822E+01 ...
  1   200         5   2.46175E+02 ... 2.42822E+01 ...
  1   250         6   2.65649E+02 ... 2.42822E+01 ...
  1   300         7   2.83120E+02 ... 2.42822E+01 ...
  1   350         8   2.95857E+02 ... 2.42822E+01 ...
  1   360  UZ     9   3.00000E+02 ... 2.42822E+01 ...
  1   400  EP    10   3.13677E+02 ... 2.42822E+01 ...
```

Restarting from label 9, *make second* is a continuation in d_2

```
BR    PT  TY  LAB    PAR(6)          PAR(11)
  1    50        11   1.22622E-02 ... 2.45827E+01 ...
  1   100        12   1.21864E-02 ... 2.46715E+01 ...
  1   150        13   1.21391E-02 ... 2.47250E+01 ...
  1   200        14   1.20405E-02 ... 2.48316E+01 ...
  1   250        15   1.15019E-02 ... 2.52655E+01 ...
  1   300        16   1.12050E-02 ... 2.53651E+01 ...
  1   350        17   1.11439E-02 ... 2.53702E+01 ...
```

```
  1   400        18   1.10751E-02 ... 2.53691E+01 ...
  1   450        19   1.08643E-02 ... 2.53162E+01 ...
  1   500        20   1.03963E-02 ... 2.48860E+01 ...
  1   550        21   1.00644E-02 ... 2.42560E+01 ...
  1   600        22   9.72560E-03 ... 2.32308E+01 ...
  1   650        23   9.51892E-03 ... 2.23003E+01 ...
  1   674  LP    24   9.51660E-03 ... 2.22876E+01 ...
  1   700  EP    25   9.52107E-03 ... 2.23120E+01 ...
```

gives a limit point. Restarting from the limit point at label 24 the two-dimensional continuation in (d_1, d_2) is prepared with *make third*

```
 BR    PT  TY  LAB    PAR(5)          PAR(6)
  2     1        26   2.50000E-01 ... 9.51660E-03 ...
  2     2        27   2.50000E-01 ... 9.51660E-03 ...
  2     3        28   2.50000E-01 ... 9.51660E-03 ...
  2     4        29   2.50000E-01 ... 9.51660E-03 ...
  2     5  EP    30   2.50000E-01 ... 9.51660E-03 ...
```

The backward two-dimensional continuation with *make fourth*

```
 BR    PT  TY  LAB   PAR(5)          PAR(6)          PAR(22)
  2    50        31  2.46219E-01 ... 9.43663E-03 ... -3.34269E-02 ...
  2   100        32  2.41461E-01 ... 9.33920E-03 ... -2.78648E-02 ...
  2   150        33  2.37038E-01 ... 9.25155E-03 ... -2.29294E-02 ...
  2   200        34  2.31438E-01 ... 9.14418E-03 ... -1.69658E-02 ...
  2   250        35  2.25278E-01 ... 9.03028E-03 ... -1.07316E-02 ...
  2   300        36  2.21512E-01 ... 8.96266E-03 ... -7.07247E-03 ...
  2   350        37  2.17917E-01 ... 8.89939E-03 ... -3.67655E-03 ...
  2   400        38  2.15545E-01 ... 8.85831E-03 ... -1.48527E-03 ...
  2   444  LP    39  2.14204E-01 ... 8.83532E-03 ... -2.63079E-04 ...
  2   491  LP    54  2.13946E-01 ... 8.83091E-03 ... -1.21572E-05 ...
  2   493  MX    55  2.13946E-01 ... 8.83091E-03 ... -1.48503E-05 ...
```

terminates around the uzer-defined $PAR(22) = 0$, where $PAR(22)$ is the log multiplier. The forward continuation *make fifth*

```
 BR    PT  TY  LAB    PAR(5)         PAR(6)           PAR(22)
  2    50        31  2.54580E-01 ... 9.61683E-03 ... -4.39433E-02 ...
  2   100        32  2.60893E-01 ... 9.76193E-03 ... -5.26539E-02 ...
  2   150        33  2.67489E-01 ... 9.92390E-03 ... -6.26690E-02 ...
  2   200        34  2.72004E-01 ... 1.00423E-02 ... -7.02137E-02 ...
  2   250        35  2.76245E-01 ... 1.01606E-02 ... -7.79540E-02 ...
  2   300        36  2.83083E-01 ... 1.03702E-02 ... -9.22711E-02 ...
  2   350        37  2.89295E-01 ... 1.05912E-02 ... -1.08339E-01 ...
```

```
2   400       38  2.95750E-01 ... 1.08812E-02 ... -1.31354E-01 ...
2   450       39  2.98916E-01 ... 1.10780E-02 ... -1.48554E-01 ...
2   500       40  3.01725E-01 ... 1.14168E-02 ... -1.82088E-01 ...
2   509  LP   41  3.01847E-01 ... 1.15004E-02 ... -1.91297E-01 ...
2   550       42  2.96838E-01 ... 1.20090E-02 ... -2.58055E-01 ...
2   600       43  2.79891E-01 ... 1.25225E-02 ... -3.50221E-01 ...
2   650       44  2.05997E-01 ... 1.35650E-02 ... -5.53043E-01 ...
2   700       45  1.77309E-01 ... 1.39176E-02 ... -3.30981E-01 ...
2   723  MX   46  1.71221E-01 ... 1.40351E-02 ... -3.06717E-05 ...
```

Observe that the LP is a turning point of the two-dimensional continuation curve, and not a real bifurcation point. The resulting curve is depicted in Figure 4.

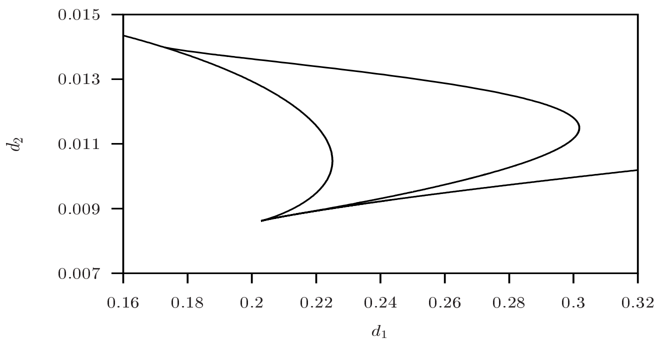

Figure 4: Two-parameter bifurcation diagram of the food chain model as detected. The end-points of the connection continuation curve are indicated by the log multiplier equal to zero, which corresponds to the points where the curve terminates at the tangent bifurcation for the limit cycle.

3 Acknowledgments

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4 Bibliography

References

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Kuznetsov, Yu. A. and Rinaldi, S. (1996). Remarks on food chain dynamics. *Mathematical Biosciences*, 134:1–33.

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5 Boundary value problem sets

The following BVP sets from the paper are added here for easy reference.

BVP for the equilibrium and related BC's

$$\left\{ \begin{array}{rcl} f(\xi,\alpha) & = & 0 \,, \\ f_\xi^{\mathrm{T}}(\xi,\alpha)v - \lambda_s v & = & 0 \,, \\ \langle v, v\rangle - 1 & = & 0 \,. \end{array} \right. \tag{3}$$

When $\dim W_-^u = 2$, as in the food chain model, a slight modification of the used homotopy method is necessary. We here have the explicit boundary conditions

$$u(0) - \xi - \varepsilon(c_1 v^{(1)} + c_2 v^{(2)}) = 0, \tag{4a}$$

$$c_1^2 + c_2^2 = 1, \tag{4b}$$

The BVP for the first homotopy step

$$\dot{x}^+ - T^+ f(x^+,\alpha) = 0, \tag{5a}$$

$$x^+(0) - x^+(1) = 0, \tag{5b}$$

$$\Psi[x^+] = 0, \tag{5c}$$

$$\dot{w} + T^+ f_u^{\mathrm{T}}(x^+,\alpha)w + \lambda w = 0, \tag{5d}$$

$$w(1) - sw(0) = 0, \tag{5e}$$

$$\langle w(0), w(0)\rangle - 1 = 0, \tag{5f}$$

$$\dot{u} - Tf(u,\alpha) = 0, \tag{5g}$$

$$\langle f(x^+(0),\alpha), u(1) - x^+(0)\rangle - h_1 = 0, \tag{5h}$$

The BVP for the second homotopy step

$$\dot{x}^+ - T^+ f(x^+,\alpha) = 0, \tag{6a}$$

$$x^+(0) - x^+(1) = 0, \tag{6b}$$

$$\langle w(0), u(1) - x^+(0)\rangle - h_2 = 0, \tag{6c}$$

$$\dot{w} + T^+ f_u^{\mathrm{T}}(x^+,\alpha)w + \lambda w = 0, \tag{6d}$$

$$w(1) - sw(0) = 0, \tag{6e}$$

$$\langle w(0), w(0)\rangle - 1 = 0, \tag{6f}$$

$$\dot{u} - Tf(u,\alpha) = 0, \tag{6g}$$

$$\langle f(x^+(0),\alpha), u(1) - x^+(0)\rangle = 0, \tag{6h}$$