CONTINUATION OF CONNECTING ORBITS IN 3D-ODES: (I) POINT-TO-CYCLE CONNECTIONS

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Abstract

This AUTO demo shows the use of a set of boundary conditions for the localization and continuation of the transcritical bifurcation of a limit cycle in a four-dimensional Marr-Pirt model.

Keywords: boundary value problems, limit cycle bifurcation, projection boundary conditions

Table 1: Parameters and variables of the Marr-Pirt model; h = hour time, mg = milligram biomass, dm = volume.

Parameter	Value	Parameter	Value
D	- h^{-1}	X_r	- $mg\ dm^{-3}$
C_1	0.4	K_1	8. $mg\ dm^{-3}$
C_2	0.6	K_2	9. $mg\ dm^{-3}$
C_3	0.6	K_3	10. $mg\ dm^{-3}$
A_1	1.25 h^{-1}	M_1	0.025 h^{-1}
A_2	0.333 h^{-1}	M_2	0.01 h^{-1}
A_3	0.25 h^{-1}	M_3	0.0075 h^{-1}

1 Marr-Pirt Model

The model we analyze here, uzing AUTO07P (see Doedel et al., 1997), is a four-dimensional food chain model known as the Marr-Pirt model. The equations are given as

$$\frac{dX_0}{dt} = \big(X_r - X_0\big) - \frac{A_1 X_0 X_1}{K_1 + X_0} \ , \tag{1a}$$

$$\frac{dX_1}{dt} = \frac{C_1 A_1 X_0 X_1}{K_1 + X_0} - M_1 X_1 - D X_1 - \frac{A_2 X_1 X_2}{K_2 + X_1} \ , \tag{1b}$$

$$\frac{dX_2}{dt} = \frac{C_2 A_2 X_1 X_2}{K_2 + X_1} - M_2 X_2 - D X_2 - \frac{A_3 X_2 X_3}{K_3 + X_2} \ , \tag{1c}$$

$$\frac{dX_3}{dt} = \frac{C_3 A_3 X_2 X_3}{K_3 + X_2} - M_3 X_3 - D X_3 \ , \tag{1d}$$

where X_i, $i = 0, 1, 2, 3$, are the variables. The parameters are described in Table 1.

The local bifurcation diagram is described in detail by Boer et al. (1998). We give a short overview here, where we focus on the parameter region $50 \leq X_r \leq 250$, $0.05 \leq D \leq 0.1$.

All local bifurcations that can be found with AUTO by default are displayed in Figure 1. The invasion criterion for X_3 is given by the transcritical bifurcation TC_e. The Hopf bifurcation H_0 indicated limit cycles of the positive equilibrium where $X_3 = 0$. The Hopf bifurcation H^+ indicates the presence of a stable equilibrium with an unstable limit cycle, which functions as a seperatrix. These bifurcations all originate at the organizing center, point M.

There are several other local bifurcations not associated with the orginizing center M. There is a tangent bifurcation of two unstable equilibria T_e, that terminates at the curve TC_e at point K. The Hopf bifurcation H^- is born from an equilibrium where all four variables are positive, and gives rise to a stable limit cycle. This limit cycle collides with the limit cycle from the Hopf bifurcation H^+ in a tangent bifurcation, T_c.

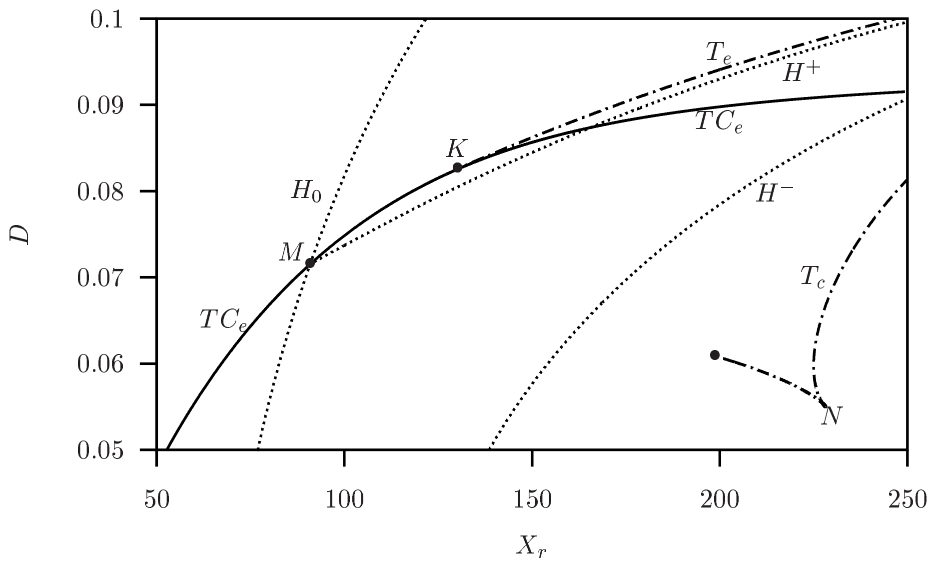

Figure 1: Local bifurcation diagram of the four-dimensional Marr-Pirt model found using AUTO07p.

Missing here is a bifurcation curve that connects the organizing center M with the tangent T_c. This curve, a transcritical bifurcation for the limit cycle, cannot be found with standard detection faciities in AUTO07p.

2 Transcritical bifurcation of a cycle

We introduce a set of boundary conditions for a limit cycle

$$\begin{cases} \dot{x}^{\pm} - f(x^{\pm}, \alpha) & = 0 \, , \\ x^{\pm}(0) - x^{\pm}(T^{\pm}) & = 0 \, , \end{cases} \tag{2a}$$

which has no unique solution yet. For that, we also introduce an integral condition. This system is capable of continuation of a limit cycle, *and* detection of the transcritical bifurcation of a limit cycle.

In the demo subdirectory *01Cycle* an appropriate starting file is generated, using default AUTO facilities. The command *make* gives the output

```
  BR    PT  TY  LAB    PAR(2)
   1     1  EP    1   1.00000E+02 ...
   1   475  HB    2   1.45704E+02 ...
   1   500  EP    3   1.48149E+02 ...
```

which gives a Hopf bifurcation, and

```
  BR    PT  TY  LAB    PAR(2)           PERIOD
   2   349  UZ    4   1.75000E+02 ... 1.11079E+02
   2   500  EP    5   1.89207E+02 ... 1.09414E+02
```

where the cycle is continued up to a uzerpoint.

The output is transported to the next subdirectory *02ContTCc*, as the file *s.C5*, where the BVP is used for detection and continuation of the transcritical bifurcation of the limit cycle. The command *make* gives the output

```
BR    PT  TY  LAB    PAR(2)           PAR(11)
 1    50        2   2.13368E+02 ... 1.01030E+02 ...
 1   100        3   2.29916E+02 ... 8.02091E+01 ...
 1   150        4   2.29729E+02 ... 7.79030E+01 ...
 1   200        5   2.43249E+02 ... 8.69430E+01 ...
 1   246  BP    6   2.63503E+02 ... 9.93303E+01 ...
 1   250  EP    7   2.65509E+02 ... 1.00512E+02 ...
```

which detects a BP, where the maximum of $X_3 = 0$. Restarting at label 6 starts up the continuation in two parameter dimensions

```
BR    PT  TY  LAB    PAR(1)           PAR(11)        PAR(2)
 2     1        8   5.50000E-02 ... 9.93303E+01   2.63503E+02 ...
 2     2        9   5.50000E-02 ... 9.93303E+01   2.63503E+02 ...
 2     3       10   5.50000E-02 ... 9.93303E+01   2.63503E+02 ...
 2     4       11   5.50000E-02 ... 9.93303E+01   2.63503E+02 ...
 2     5  EP   12   5.50000E-02 ... 9.93303E+01   2.63503E+02 ...
```

and

```
BR    PT  TY  LAB    PAR(1)           PAR(11)        PAR(2)
 2    10  BP   14   5.49294E-02 ... 9.96339E+01   2.64373E+02 ...
 2    20  BP   17   5.44050E-02 ... 1.01917E+02   2.70907E+02 ...
 2    30  BP   20   5.38728E-02 ... 1.04285E+02   2.77673E+02 ...
 2    40  BP   23   5.33514E-02 ... 1.06658E+02   2.84437E+02 ...
 2    50  EP   26   5.28895E-02 ... 1.08805E+02   2.90544E+02 ...
```

Obviously this last run can be extented. The resulting curve TC_c is depicted in Figure 2.

3 Acknowledgements

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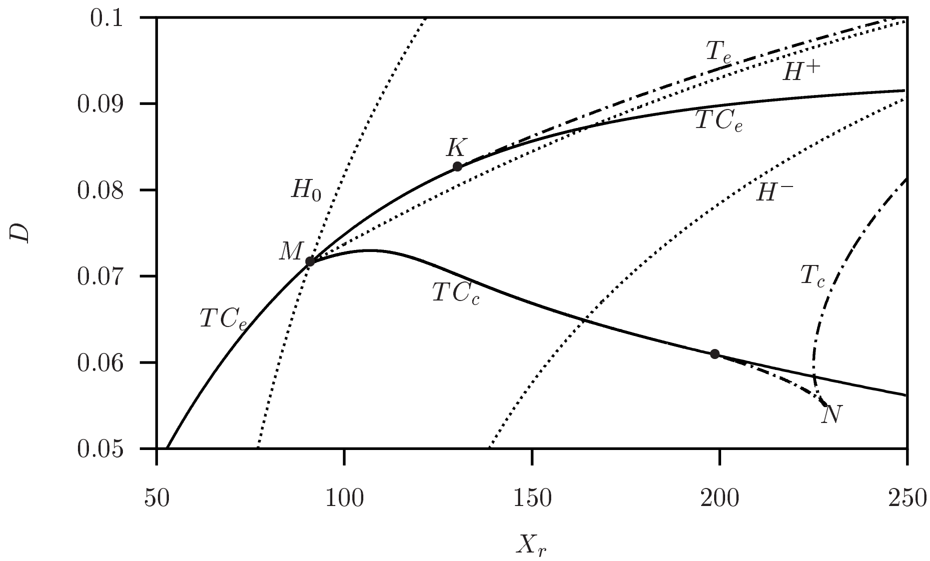

Figure 2: Local bifurcation diagram of the four-dimensional Marr-Pirt model, including the transcritical bifurcation for a limit cycle TC_c, that originates in the organizing center M.