On the Mechanics of the Modern Working-Recurve Bow [1]

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Abstract

Characteristic for the bow are the slender elastic arms or limbs. The bow is braced by putting a string shorter than the bow between the tips of the limbs. Additional deformation energy is stored in the elastic limbs by drawing the bow into the fully drawn position. Part of this amount of energy is transformed into kinetic energy of a light arrow.

In the 1930's the design of the bow became a subject of scientific research. Experiments were performed in which design parameters were changed more or less systematically. However, the mathematical models were rather simple. Because fast computers are now available the presented model in this paper can be much more advanced. The resulting set of partial differential equations with known initial values and moving boundaries is solved numerically using a finite-difference method.

In this paper the design parameters associated with the developed model are charted accurately. Bows used in the past and nowadays on shooting meetings such as the Olympic Games are compared. It turns out that the application of better materials which can store more deformation energy per unit of mass and that this material is used to a larger extent, contribute most to the improvement of the bow. The parameters which fix the mechanical performance of the bow appear to be less important as is often claimed.

[1] *Computational Mechanics* **8**: 291-304 (1991)

1 Introduction

In the 1930's bows and arrows became the object of study by scientists and engineers, Hickman [3] and Klopsteg [4]. Their work influenced strongly the design and construction of the bow and arrow. Experiments were performed to determine the influence of different parameters. They also made mathematical models. As part of modelling simplifying assumptions had to be made in order to obtain a solution in closed form or to approximate the solution of the governing equations numerically in an acceptable amount of computing time. Because of these simplifications only bows with specific features could be described.

In Kooi and Sparenberg [9] we dealt with the statics of the so called working-recurve bows. In this paper we consider the dynamics of this type of bow. These developed mathematical models are much more advanced, so that more detailed information was obtained giving a better understanding of the action of rather general types of bow.

In Section 2 of this paper all design parameters are charted accurately and quality coefficients are identified. The governing set of partial differential equations with known initial values and moving boundaries are derived in Section 3. The equations of equilibrium are derived through the use of variational principles. The resulting equations constitute a free-boundary value problem for a set of ordinary differential equations. In Kooi and Sparenberg [9] a shooting method was derived to solve this set of equations numerically. The equations of motions form a moving-boundary value problem for a set of partial differential equations. The subject of Section 4 is a description of the developed numerical method. A finite-difference technique with complete discretization (both the space and time variables are written in difference form) is applied. The resulting moving-boundary problem is solved using a front-tracking method with a fixed grid.

In Section 5 the performance of different types of bow are compared. Roughly speaking the design parameters can be divided into two groups. One determines the mechanical performance of the bow. Within certain limits, these parameters appear to be less important as is often claimed. The other group of parameters concerns the strength of the materials and the way these materials are used in the construction of the bow. It turns out that the application of better materials and that more of this material is used to a larger extent, contribute most to the improvement of the bow.

2 Formulation of the problem

In essence the bow proper consists of two elastic limbs, often separated by a rigid middle part called grip. The bow is braced by fastening a string between both ends of the limbs. The distance between the grip on the belly side and the string in that situation is called the brace height or fistmele. After an arrow is set on the string the archer pulls the bow from braced situation in full draw. Then, after aiming, the arrow is loosed or released. The force in the string accelerates the arrow and transfers part of the stored potential energy in the elastic parts of the bow, as kinetic energy into the arrow. Meanwhile the bow is held in its place and the archer feels a recoil force in the bow hand. The velocity of the

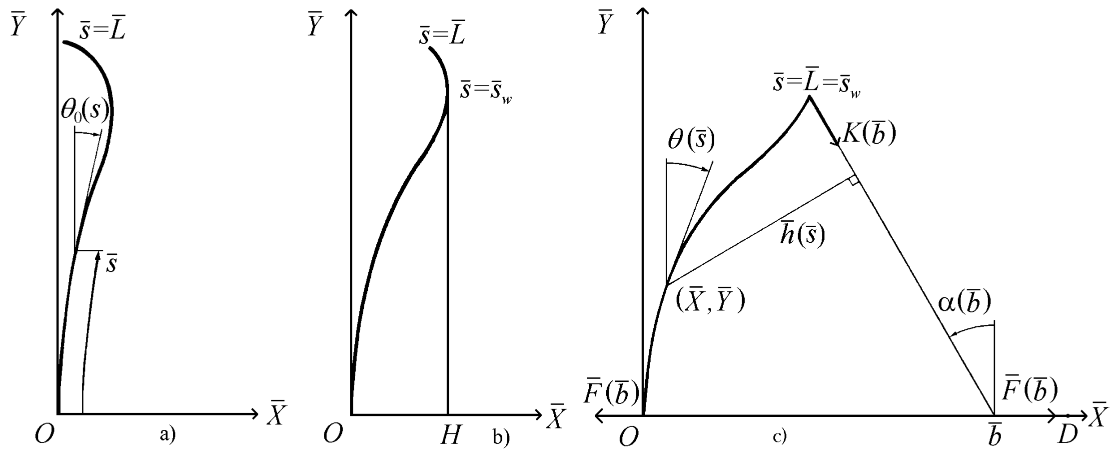


Figure 1: Three situations of the working-recurve bow: (a) unbraced, (b) braced, (c) partly drawn.

arrow when leaving the string fixed by the moment the acceleration of the arrow is zero, is called the muzzle velocity or initial velocity.

We are concerned with bows of which the limbs move in a flat plane, and which are symmetric with respect to the line of aim. The arrow will pass through the midpoint of the bow as in the case of a "center-shot bow". The bow is placed in a Cartesian coordinate system $(\overline{x}, \overline{y})$, the line of symmetry coinciding with the $\overline{x}$-axis and the origin O coinciding with the midpoint of the bow. We assume the limbs to be inextensible and that the Euler-Bernoulli beam theory holds. The total length of the bow is denoted by $2\overline{L}$. In our theory it will be represented by an elastic line of zero thickness, along which we have a length coordinate $\overline{s}$ measured from O, hence for the upperhalf we have $0 \leq \overline{s} \leq \overline{L}$. This elastic line is endowed with bending stiffness $\overline{W}(\overline{s})$ and mass per unit of length $\overline{V}(\overline{s})$. In Figure 1(a) the unbraced situation (without string) is shown. With a working-recurve bow the limbs are curved in the 'opposite' direction in the unstrung situation. The geometry of the bow is described by the local angle $\theta_0(\overline{s})$ between the elastic line and the $\overline{y}$-axis, the subscript 0 indicates the unstrung situation. $\overline{L}_0$ is the half length and $2\overline{m}_g$ the mass of the grip.

In Figure 1(b) the working recurve bow is braced by applying a string. Because of the shape of the unstrung bow the string lies along the bow near the tips with coordinates $(\overline{x}_t, \overline{y}_t)$. There may be concentrated masses $\overline{m}_t$ with moment of inertia $\overline{J}_t$ at each of the tips, representing for instance horns used to fasten the string. The length of the unloaded string is denoted by $2\overline{l}_0$, its mass by $2\overline{m}_s$. We assume that the material of the string obeys Hooke's law, the strain stiffness is denoted by $\overline{U}_s$. Note that whether the length of the string or the brace height denoted by $|\overline{OH}|$ fixes the shape if the bow in braced situation.

Of a working-recurve bow the parts near the tips are elastic and bend during the final part of the draw. When drawing such a bow the length of contact between string and limb decreases gradually until the point where the string leaves the limb, denoted by $\overline{s} = \overline{s}_w$, coincides with the tip $\overline{s} = \overline{L}$ and remains there during the final part of the draw. We assume that there is no friction between bow and string for $\overline{s}_w \leq \overline{s} \leq \overline{L}$. Most modern bows are working-recurve bows. In Figure 1(c) the bow is pulled by the force $\overline{F}(\overline{b})$ into a partly drawn position where the middle of the string has the $\overline{x}$-coordinate $\overline{b}$. To each bow belongs a value $\overline{b} = |\overline{OD}|$ for which it is called fully drawn indicated by a subscript 1. The force $\overline{F}(|\overline{OD}|)$ is called the weight of the bow and the distance $|\overline{OD}|$ is its draw. By releasing the drawn string at time $\overline{t} = 0$ and holding the bow at its place, the arrow, represented by a point mass $2\overline{m}_a$ is propelled. During the acceleration at some moment $\overline{t} = \overline{t}_b$ the string touches the belly side of the limb of a working-recurve bow again. The arrow leaves the string when the acceleration of the midpoint of the string becomes negative. This moment is denoted by $\overline{t}_l$ and the muzzle velocity of the arrow is denoted by $\overline{c}_l$.

A shorthand notation for a bow and arrow combination is introduced with

$$\begin{aligned}\overline{B}(\overline{L}, \overline{L}_0, \overline{W}(\overline{s}), \overline{V}(\overline{s}), \theta_0(\overline{s}), \overline{m}_a, \overline{m}_t, \overline{J}_t, \overline{m}_e, \overline{J}_e, \overline{m}_g, \overline{U}_s, \overline{m}_s, |\overline{OH}| \\ \text{or } \ \overline{l}_0; |\overline{OD}|, \overline{F}(|\overline{OD}|), \overline{m}_b) \ ,\end{aligned} \tag{1}$$

where $\overline{m}_b$ is the mass of one limb excluding the mass of the grip. In an input-output philosophy all parameters are input variables which determine the mechanical action of the bow and arrow combination. In this notation $\overline{B}$ stands for the set of output variables in which one is interested. We will give examples later on when we introduce quality qoefficients. Note that the last two mentioned parameters are added to the list artificially. This implies that both functions $\overline{W}(\overline{s})$ and $\overline{V}(\overline{s})$ are constrained. We consider the values of these functions for $\overline{s} = \overline{L}_0$ to be already fixed by both constraints. The first constraint concerning the weight, is an implicit relationship between a number of parameters of which $\overline{W}(\overline{s})$ is one of them, and the weight $\overline{F}(|\overline{OD}|)$ of the bow. The second constraint is just

$$\overline{m}_b = \int_{\overline{L}_0}^{\overline{L}} \overline{V}(\overline{s}) d\overline{s} \ . \tag{2}$$

and for a given mass of the bow the value $\overline{V}(\overline{L}_0)$ is derived with easy. This shows that both functions are considered to be the product of a function $\overline{W}(\overline{s})/\overline{W}(\overline{L}_0)$ and $\overline{V}(\overline{s})/\overline{V}(\overline{L}_0)$ of the length coordinate $\overline{s}$ into IR and a parameter $\overline{W}(\overline{L}_0)$ and $\overline{V}(\overline{L}_0)$ with dimensions. These resulting functions together with the function $\theta_0(\overline{s})$ make the bow a distributed parameter structure.

The quantities $|\overline{OD}|, \overline{F}(|\overline{OD}|), \overline{m}_b$ are taken as elements of a dimensional base in a dimensional analysis. The use of the dimensional analysis technique gives, when $\overline{K}$ equals the force in the string

$$\overline{L} = L \, |\overline{OD}| \ , \quad \overline{K} = K \, \overline{F}(|\overline{OD}|) \ , \quad \overline{m}_a = m_a \, \overline{m}_b \ , \tag{3}$$

and for the functions of the length coordinate s

$$\overline{W}(\overline{s}) = W(s) |\overline{OD}|^2 \cdot \overline{F}(|\overline{OD}|) \ , \quad \overline{V}(\overline{s}) = V(s) \frac{\overline{m}_b}{|\overline{OD}|} \ . \tag{4}$$

Observe that these functions of $\overline{s}$ are also transformed to functions of the dimensionless length coordinate s. Also the angle $\theta(\overline{s})$ between the elastic line and the $\overline{y}$-axis will be transformed to $\theta(s)$, where we should have used a new symbol. With respect to dimensional analysis this yields no added difficulties. Finally we have

$$|\overline{OD}| = |OD| \text{ cm} \; , \quad \overline{F}(|\overline{OD}|) = F(|OD|) \text{ kgf} \; , \quad \overline{m}_b = m_b \text{ kg} \; . \tag{5}$$

So, quantities with dimension are labelled by means of a bar $-$ and quantities without the underscore are the associated dimensionless quantities. The unit of time is already fixed by the choice of the other 3 units: cm, kgf and kg. For the time $\overline{t}_l$, the moment the arrow leaves the string, we have for instance

$$\overline{t}_l = t_l(L, \cdots, |OH| \text{ or } l_0) \cdot \sqrt{\frac{\overline{m}_b \cdot |\overline{OD}|}{\overline{F}(|\overline{OD}|)}} = t_l \cdot \sqrt{\frac{m_b \cdot |OD|}{F(|OD|)}} \cdot \sqrt{\frac{\text{kg} \cdot \text{cm}}{\text{kgf}}} \; . \tag{6}$$

This means that the unit of time equals 0.03193 sec.

We introduce a number of quality coefficients which can be used to judge the performance of a bow and arrow combination. The static quality coefficient q is given by

$$q = \frac{\overline{A}}{|\overline{OD}| \cdot \overline{F}(|\overline{OD}|)} \; , \tag{7}$$

so equal to the dimensionless energy stored in the elastic parts of the bow, the working parts of the limb and the string, by deforming the bow from the braced position into the fully drawn position. This quantity is given by

$$\overline{A} = \int_{\overline{b}=|\overline{OH}|}^{\overline{b}=|\overline{OD}|} \overline{F}(\overline{b}) d\overline{b} \; . \tag{8}$$

One of the dynamic quality coefficients is the efficiency η defined by

$$\eta = \frac{\overline{m}_a \cdot \overline{c}_l^2}{\overline{A}} \; , \tag{9}$$

where $\overline{c}_l$ is the muzzle velocity. The efficiency is by definition dimensionless. The second dynamic quality coefficient is the dimensionless version of the muzzle velocity denoted by ν,

$$\nu = \sqrt{\frac{q \cdot \eta}{m_a}} \; . \tag{10}$$

So, it is just a combination of the other two coefficients and the mass of the arrow. Observe that by definition these quality coefficients are dimensionless. This means that the sensitivities of these coefficients with respect to the elements of the dimensional base $|\overline{OD}|, \overline{F}(|\overline{OD}|)$ and $\overline{m}_b$ can be obtained directly, without solving the governing equations of motion which constitute the mathematical model again. The advantage of this technique is that with the comparison of different bows, taking the quantities $|OD|, F(|OD|)$ and m_b equal to 1, yields interpretable results for the quality coefficients.

3 Mathematical modelling

3.1 Equations of equilibrium

In Kooi and Sparenberg [9] we wrote down the governing equations in the various static situations. The unknown variables were $\theta(s), x(s)$ and $y(s)$ as shown in Figure 1. In this section we derive the equations of equilibrium again. However, we now start from the unknown functions $x(s), y(s)$, the bending moment $M(s)$ and the normal force $T(s)$. Further we shall apply variational principles to obtain the balance equations.

We assume that the part of the limb in contact with the string $s_w \leq s \leq L$, possesses the undeformed shape. This is because the string lies along the neutral line of the limb, so that no bending moment or shear force can be originated from the force in the string. Then the total potential energy in one limb and half string, equals

$$\frac{1}{2}A(b) = \int_{L_0}^{s_w} \frac{1}{2}W(s) \cdot \big((x'y'' - y'x'') + \theta_0'\big)^2 ds + \frac{1}{2}U_s \cdot \frac{(l-l_0)^2}{l_0} , \tag{11}$$

where x' denotes $\frac{dx}{ds}$, etc.

$$L - s_w + \big(b - x(s_w)^2 + y(s_w)^2\big)^{1/2} = l . \tag{12}$$

Further we have the constraint

$$x'^2 + y'^2 = 1 , \quad L_0 \leq s \leq L , \tag{13}$$

and for $s = L_0$ the geometric boundary conditions

$$x = x_0 , \quad y = y_0 , \quad y_0'(L_0) \cdot x'(L_0) = x_0'(L_0) \cdot y'(L_0) . \tag{14}$$

We define the following two functions

$$G(s) = \frac{1}{2}W(s) \cdot (x'y'' - y'x'' + \theta_0')^2 - \lambda(s)\big((x'^2 + y'^2 - 1)\big) , \tag{15}$$

and

$$H(s_w) = \frac{1}{2}U_s \cdot \frac{L - s_w + \big(b - x(s_w)^2 + y(s_w)^2\big)^{1/2} - l_0}{l_0} , \tag{16}$$

where $\lambda(s)$ is an unknown Lagrangian multiplier to meet the constraint (13). Further we define

$$\Lambda = \int_{L_0}^{s_w} G(s, x', y', x'', y'')ds + H(s, x, y)|^{s=s_w} . \tag{17}$$

The principle of the stationary potential energy states that the state of equilibrium is characterized by

$$\delta\Lambda = 0 , \tag{18}$$

for all admittable configurations, thus obeying (14). By calculus of variation, see for instance Gelfand and Fomin [2], we obtain the equations of balance

$$(Tx')' + (M'y')' = 0 \ , \quad L_0 \le s \le s_w \ , \tag{19}$$

and

$$(Ty')' - (M'x')' = 0 \ , \quad L_0 \le s \le s_w \ . \tag{20}$$

In these equations $M(s)$ is the bending moment according to Euler-Bernoulli,

$$M(s) = W(s) \cdot (x'y'' - y'x'' + \theta_0') \ , \quad L_0 \le s \le s_w \ , \tag{21}$$

and $T(s)$ is the normal force, connected to the Lagrangian multiplier $\lambda(s)$ by

$$\lambda(s_w) = -\frac{1}{2}T(s) + \frac{M(s)}{W(s)} \cdot \big(M(s) - W(s)\theta_0'(s)\big) \ , \quad L_0 \le s \le s_w \ , \tag{22}$$

Further we have 5 free-boundary conditions at $s = s_w$,

$$G_{x'} - \frac{d}{ds}G_{x''} + H_x = 0 \ , \tag{23}$$

$$G_{x''} = 0 \ , \tag{24}$$

$$G_{y'} - \frac{d}{ds}G_{y''} + H_y = 0 \ , \tag{25}$$

$$G_{y''} = 0 \ , \tag{26}$$

$$H_s + G - (G_{x'} - \frac{d}{ds}G_{x''})x' - (G_{y'} - \frac{d}{ds}G_{y''})y' - G_{x''}x'' - G_{y''}y'' = 0 \ , \tag{27}$$

where $G_{x'}$ denotes $\frac{\partial G}{\partial x'}$, etc. The conditions (24) and (26) read

$$-y'M = x'M = 0 \ , \quad s = s_w \ , \tag{28}$$

and because of (13), this implies

$$M(s_w) = 0 \ , \tag{29}$$

Using (29) and (21) the conditions (23) and (25) read

$$-Tx' - y'M' + K\sin\alpha = 0 \ , \quad s = s_w \ , \tag{30}$$

$$Ty' - x'M' + K\cos\alpha = 0\ , \quad s = s_w\ , \tag{31}$$

respectively, where K is the force in the string

$$K = U_s \cdot \frac{L - s_w + \big(b - x(s_w)^2 + y(s_w)^2\big)^{1/2} - l_0}{l_0}\ , \tag{32}$$

and α the angle between the string and the y-axis, reckoned positive in the direction indicated in Figure 1(c), thus

$$\sin\alpha = \frac{b - x(s_w)}{\big(b - x(s_w))^2 + (y(s_w))^2\big)^{1/2}}\ , \tag{33}$$

and

$$\cos\alpha = \frac{y(s_w)}{\big(b - x(s_w))^2 + (y(s_w))^2\big)^{1/2}}\ . \tag{34}$$

Equation (27) becomes after some formula manipulation

$$K + T(s_w) = 0\ . \tag{35}$$

Substitution of (35) into (30) and (31) gives

$$-(x' + \sin\alpha)T - y'M' = 0\ , \quad s = s_w\ , \tag{36}$$

$$(y' - \cos\alpha)T - x'M' = 0\ , \quad s = s_w\ . \tag{37}$$

Using (35) again and the fact that $T \neq 0$, we obtain

$$M'(s_w) = 0\ , \tag{38}$$

$$x'(s_w)y(s_w) = -\big(b - x(s_w)\big)y'(s_w)\ . \tag{39}$$

The resulting boundary conditions at $s = s_w$ are (29), (35), (38) and (39) together with condition (32). In the next section we use these relations to derive the equations of motion.

For each b the drawing force denoted by $F(b)$ is given by

$$F(b) = 2K(b)\sin\alpha(b)\ . \tag{40}$$

In Kooi and Sparenberg [9] we solved for $|OH| \leq b \leq |OD|$ the free-boundary value problem using a shooting method. The solution for each value of b is expressed in the functions $\theta(s), x(s), y(s)$ and K or F, but the calculation of $M(s)$ and $T(s)$ is straightforward, just as to the shear force $Q(s)$, given by

$$Q(s) = -M'(s)\ , \quad L_0 \leq s \leq L\ . \tag{41}$$

The results for $b = |OD|$ are used as an initial guess in the procedure for the solution of the static finite-difference scheme given in Section 4.

3.2 Equations of motion

In this section the governing equations of motion for the symmetric working-recurve bow are given. A simple lumped parameter model for the string is used. The mass of the string is accounted for by placing one third of the mass of the string $2m_s$ at the end of string where it fits in the nock of the limb and one sixth (because of symmetry) at the other end, where the arrow contacts the string at nocking point. Further, it is assumed that the elastic string slides along the inextensible limb without friction.

Generally the string clears the limb on the fully drawn bow. Then, for $t = 0$ when the arrow is released, we have $s_w = L$. For $0 \leq t \leq t_b$ the equations of motion are those given in Kooi [5]. The equations read

$$V\ddot{x} = (Tx')' + (M'y')' \;, \quad L_0 \leq s \leq L \;,\; t \geq 0 \;, \tag{42}$$

and

$$V\ddot{y} = (Ty')' - (M'y')' \;, \quad L_0 \leq s \leq L \;,\; t \geq 0 \;, \tag{43}$$

where x' denotes $\frac{\partial x}{\partial s}$, and $\ddot{x}$ denotes $\frac{\partial^2 x}{\partial t^2}$, etc. For the whole limb we have

$$x'^2 + y'^2 = 1 \;, \quad L_0 \leq s \leq L \;,\; t \geq 0 \;, \tag{44}$$

and

$$M(s) = W(s) \cdot (x'y'' - y'x'' + \theta_0') \;, \quad L_0 \leq s \leq L \;,\; t \geq 0 \;. \tag{45}$$

The geometric boundary conditions for $s = L_0$ and $t \geq 0$ read

$$x(L_0, t) = x_0(L_0) \;, \quad y(L_0, t) = y_0(L_0) \;, \quad y_0'(L_0) \cdot x'(L_0, t) = x_0'(L_0) \cdot y'(L_0, t) \;, \tag{46}$$

and for $s = L$ and $t \geq 0$, with $J_t = 0$

$$M(L, t) = 0 \;. \tag{47}$$

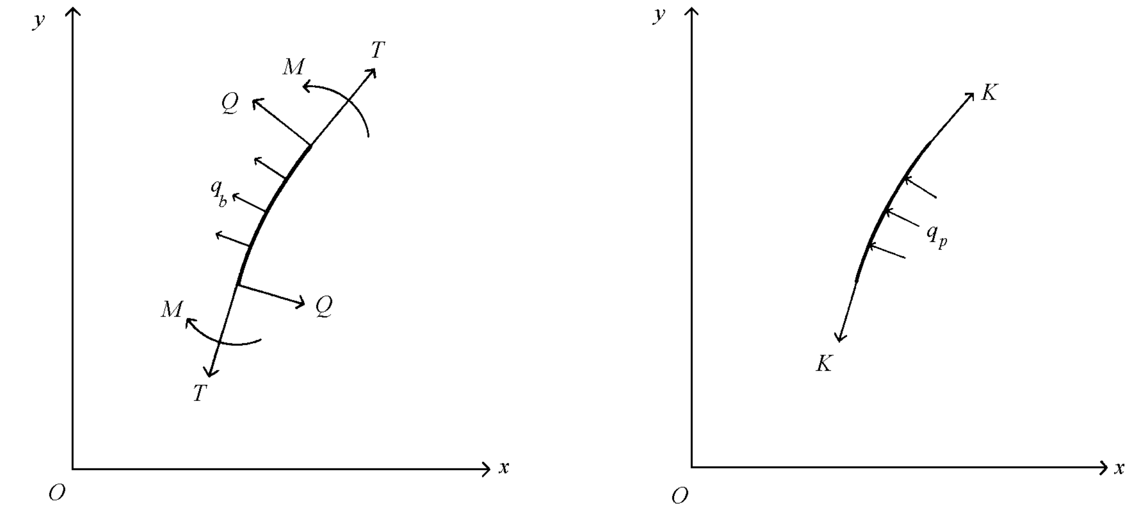
For $s = L$ and $t \leq t_b$ the equation for the length of the string is

$$l(t) = \left((b - x(L,t))^2 + (y(L,t))^2\right)^{1/2} \;, \tag{48}$$

and the equations of motion for the concentrated mass m_t at the tip

$$(\frac{2}{3}\, m_s + m_t)\ddot{x}(L,t) = -x'T - y'M' + K(t) \cdot \frac{b(t) - x(L,t)}{l(t)} \;, \quad s = L \;, \tag{49}$$

$$-(\frac{2}{3}\, m_s + m_t)\ddot{y}(L,t) = y'T - x'M' + K(t) \cdot \frac{y(L,t)}{l(t)} \;, \quad s = L \;, \tag{50}$$



a) b)

Figure 2: Forces and moments acting upon an element (a) of the limb and (b) of the string.

in the x- and y-direction, respectively. The string force K for $t \geq 0$ is given by

$$K(t) = U_s \cdot \frac{l(t) - l_0}{l_0} \,, \tag{51}$$

and the equation of motion for the arrow, which completes the governing set of equations, is for $0 \leq t \leq t_b$

$$\dot{b} = c \quad , \quad (m_a + \frac{1}{3} m_s)\dot{c} = -K(t) \cdot \frac{b(t) - x(L,t)}{l(t)} \,, \tag{52}$$

where $K(t)$ is given by (51) and $l(t)$ by (48). For $t_l \leq t$ we have $m_a = 0$, because the arrow left the string.

The time t_b is fixed by the condition

$$x'(L,t)y(L,t) + \big(b - x(L,t)\big)y'(L,t) = 0 \,. \tag{53}$$

We now derive the equation of motion for $t \geq t_b$. In Figure 2(a) the resultant forces and moments acting upon a differential element of the limb in contact with the string are shown. The momentum balance in the x- and y-direction yield

$$V\ddot{x} = -q_b y' + (Tx')' + (M'y')' \,, \quad s_w \leq s \leq L \,,\; t \geq t_b \,, \tag{54}$$

and

$$V\ddot{y} = q_b x' + (Ty')' - (M'y')' \,, \quad s_w \leq s \leq L \,,\; t \geq t_b \,, \tag{55}$$

where q_b is the force of the string acting upon the limb, per unit of length along the limb. The equations of motion for the part in contact with the limb, see Figure 2(b), read

$$0 = -q_p y' + (Kx')' \,, \quad 0 \leq r \leq r_w \,,\; t \geq t_b \,, \tag{56}$$

and

$$0 = q_p x' + (Ky')' \ , \quad 0 \le r \le r_w \ , \ t \ge t_b \ , \tag{57}$$

where q_p is the force of the limb acting upon the string, per unit of length long the limb and r is the length coordinate along the elastic string measured form the loop. This coordinate equals r_w at the place where the string leaves the limb. Because of the lumped parameter model for the string the left-hand side of these equations are zero: the equations are quasi-static. We have

$$q_p(r) = -q_b(s) \ , \quad r = L - s \ , \quad s_w \le s \le L \ , \ 0 \le r \le r_w \ , \ t \ge t_b \ , \tag{58}$$

Equations (54), (56) and (55), (57), using (58) can be combined into two equations of motion for $s_w \le s \le L$, $t \ge t_b$,

$$V\ddot{x} = (Sx')' + (M'y')' \ , \quad L_0 \le s \le L \ , \ t \ge t_b \ , \tag{59}$$

and

$$V\ddot{y} = (Sy')' - (M'y')' \ , \quad L_0 \le s \le L \ , \ t \ge t_b \ , \tag{60}$$

with $S(s,t) = K(t) + T(s,t)$ as a new unknown function of s and t. The equations which describe the motion of the free part of the limb are just (42) and (43). Also the boundary conditions (46), (47) and (51) remain valid for $t \ge t_b$. The three conditions for $s = L$ become

$$l(t) = L - s_w + \big(b - x(s_w,t)\big)^2 + (y(s_w,t))^2\big)^{1/2} \ , \tag{61}$$

$$(\frac{2}{3}\, m_s + m_t)\ddot{x}(L,t) = -x'S - y'M' \ , \quad s = L \ , \tag{62}$$

$$-(\frac{2}{3}\, m_s + m_t)\ddot{y}(L,t) = y'S - x'M' \ , \quad s = L \ . \tag{63}$$

The conditions at the moving boundary $s = s_w$ and $t_b \le t$, are (see also equations (29), (38), (35) and (39),

$$\lim_{s\uparrow s_w} x = \lim_{s\downarrow s_w} x \ , \quad \lim_{s\uparrow s_w} x' = \lim_{s\downarrow s_w} x' \ , \tag{64}$$

$$\lim_{s\uparrow s_w} y = \lim_{s\downarrow s_w} y \ , \quad \lim_{s\uparrow s_w} y' = \lim_{s\downarrow s_w} y' \ , \tag{65}$$

$$\lim_{s\uparrow s_w} M = \lim_{s\downarrow s_w} M \ , \quad \lim_{s\uparrow s_w} M' = \lim_{s\downarrow s_w} M' \ , \tag{66}$$

$$\lim_{s \uparrow s_w} T = \lim_{s \downarrow s_w} S - K \,, \tag{67}$$

$$x'(s_w,t)y(s_w,t) + \big(b(t) - x(s_w,t)\big)y'(s_w,t) = 0 \,. \tag{68}$$

The equation of motion for the arrow for $t_b \leq t$ becomes

$$\dot{b} = c \quad , \quad (m_a + \frac{1}{3}m_s)\dot{c} = -K(t) \cdot \frac{b(t) - x(s_w,t)}{l(t) - (L - s_w)} \,. \tag{69}$$

In summary, for $0 \leq t \leq t_b$ we have just one set of partial differential equations defined on $s \in [L_0, L]$ with unknown functions x, y, M, T. For $t_b \leq t$ there are two sets of partial differential equations, defined on $s \in [L_0, s_w]$ with unknown functions x, y, M, T and $s \in [s_w, L]$ with unknown functions x, y, M and S, where s_w is the place of the moving boundary. In the latter case we have, in addition to the boundary conditions at $s = L_0$ and $s = L$, conditions at $s = s_w$, whereby s_w is unknown and has to be determined in the course of the calculation.

In addition to the displacements $x(s,t)$ and $y(s,t)$ we introduce the velocities in x- and y-direction denoted by $u(s,t)$ and $v(s,t)$, respectively. Then the initial conditions at $t = 0$ read

$$x(s,0) = x_1(s) \,, \quad y(s,0) = y_1(s) \,, \quad u(s,0) = 0 \,, \quad v(s,0) = 0 \,, \tag{70}$$

and

$$b(0) = |OD| \,, \quad c(0) = 0 \,, \tag{71}$$

where $x_1(s)$ and $y_1(s)$ fix the shape of the bow in fully drawn situation and which are solutions of the static equations given in Section 3.1.

In the next section we discuss a finite-difference method for the solution of this moving-boundary value problem.

4 Finite-difference equations

In this section the finite-difference equations for the numerical solution of the moving-boundary value problem stated in the preceding section are given. There are three different finite-difference approaches:

(a) The method of lines. Only the time variable is discretized. At successive time levels an ordinary differential equation has to be solved.

(b) The semi-finite-difference method. Only the space variable is discretized. In this way the partial differential equations are reduce to a set of coupled ordinary differential equations which can be integrated in time using, for instance, a Runge-Kutta method.

(c) The complete difference method. Both space and time variables are written in a differential form.

In this paper we propose a complete finite-difference method of the Crank-Nicolson type with front-tracking for the moving boundary, see also Crank [1].

We consider a uniform mesh along the limb

$$s = j\Delta s \; , \quad j = 0(1)n_s \; , \; n_s \Delta s = L - L_0 \; , \tag{72}$$

and

$$t = k\Delta t \; , \quad k = 0(1)n_t \; , \tag{73}$$

n_t being an integer large enough to cover the time interval of interest. The displacements x and y, the velocities u and v and the bending moment M are unknown in these gridpoints, while the normal force in the limb T is defined at each time level only at points just in between the gridpoints. In order to satisfy the boundary conditions, fictitious external mesh points are introduced $(L_0 - \Delta s, k\Delta t)$ and $(L + \Delta s, k\Delta t)$ with $k = 0(1)nt$. Furthermore two difference operators are defined

$$\delta f_{j,k} = f_{j+1/2,k} - f_{j-1/2,k} \; , \quad \Delta f_{j,k} = \delta f_{j+1/2,k} - \delta f_{j-1/2,k} \; . \tag{74}$$

If we use a weighted average of forward and backward approximation, equations (42), (44) and (45) become

$$\begin{aligned} V_j \frac{u_{j,k+1} - u_{j,k}}{\Delta t} &= \mu \Big(\frac{\delta (T\delta x)_{j,k+1}}{\Delta s^2} + \frac{\delta (\delta M \delta y)_{j,k+1}}{\Delta s^3} \Big) + \\ &(1-\mu) \Big(\frac{\delta (T\delta x)_{j,k}}{\Delta s^2} + \frac{\delta (\delta M \delta y)_{j,k}}{\Delta s^3} \Big) \; , \quad j = 0(1)n_s \; , \end{aligned} \tag{75}$$

$$\frac{x_{j,k+1} - x_{j,k}}{\Delta t} = \mu u_{j,k+1} + (1-\mu) u_{j,k} \; , \quad j = 0(1)n_s \; , \tag{76}$$

$$\Big(\frac{\Delta x_{j-1/2,k+1}}{\Delta s} \Big)^2 + \Big(\frac{\Delta y_{j-1/2,k+1}}{\Delta s} \Big)^2 = 1 \; , \quad j = 0(1)n_s + 1 \; , \tag{77}$$

and

$$M_{j,k+1} = W_j \Big(\frac{\Delta x_{j,k+1} \delta^2 y_{j,k+1} - \Delta y_{j,k+1} \delta^2 x_{j,k+1}}{2\Delta s^3} + \theta_0'(j\Delta s) \Big) \; , \quad j = 0(1)n_s \; , \tag{78}$$

where $0 \leq m \leq 1$. For $m = 0$ the method is explicit and for $m = 1$ implicit. In the Crank-Nicolson method m equals $1/2$. The geometric boundary conditions for $s = L_0$ become

$$x_{0,k+1} = x_0(L_0) \; , \quad y_{0,k+1} = y_0(L_0) \; , y_0'(L_0) \frac{\Delta x_{0,k+1}}{2\Delta s} = x_0'(L_0) \frac{\Delta y_{0,k+1}}{2\Delta s} \; , \tag{79}$$

and for $s = L$ equations (47) yields

$$M_{n_s,k+1} = 0 \ , \tag{80}$$

The equations of motion in the x-direction for the concentrated mass m_t at the tip (49) becomes

$$\begin{aligned}(\frac{2}{3}\, m_s + m_t)\frac{u_{n_s,k+1} - u_{n_s,k}}{\Delta t} &= \mu\Big(-\frac{1}{2}(T_{n_s+1/2,k+1} + T_{n_s-1/2,k+1})\frac{\Delta x_{n_s,k+1/2}}{2\Delta s} - \\ &\qquad \frac{\Delta M_{n_s,k+1}\Delta y_{n_s,k+1}}{4\Delta s^2} + K_{\cdot,k+1}\frac{b_{k+1} - x_{n_s,k+1}}{l_{\cdot,k+1}}\Big) + \\ &\qquad (1-\mu)\Big(-\frac{1}{2}(T_{n_s+1/2,k} + T_{n_s-1/2,k})\frac{\Delta x_{n_s,k}}{2\Delta s} - \\ &\qquad \frac{\Delta M_{n_s,k}\Delta y_{n_s,k}}{4\Delta s^2} + K_{\cdot,k}\frac{b_k - x_{n_s,k}}{l}\cdot, k] \ ,\end{aligned} \tag{81}$$

The equations for the y-direction originating from (43) and (50) are obtained in an analogous way. The discretized equation for the length of the string (48) becomes

$$b_{k+1} = x_{n_s,k+1} + \left(l^2_{\cdot,k+1} - y^2_{n_s,k+1}\right)^{1/2} \ . \tag{82}$$

The string force K equation (51) is given by

$$K_{\cdot,k+1} = U_s \cdot \frac{l_{\cdot,k+1} - l_0}{l_0} \ , \tag{83}$$

and the equation of motion for the arrow (52), which completes the governing set of difference equations become

$$\frac{b_{k+1} - b_k}{\Delta t} = \mu c_{k+1} + (1-\mu)c_k \ , \tag{84}$$

$$(m_a + \frac{1}{3}m_s)\frac{c_{k+1} - c_k}{\Delta t} = \mu\big(-K_{\cdot,k+1}\frac{b_{k+1} - x_{n_s,k+1}}{l_{\cdot,k+1}}\big) + (1-\mu)\big(- K_{\cdot,k}\frac{b_k - x_{n_s,k}}{l_{\cdot,k}}\big) \ . \tag{85}$$

At $t = 0, b(0) = |OD|$, the solution obtained with the shooting method as described in Kooi and Sparenberg [9] is adapted using this finite-difference scheme (with all inertia term equal zero). The dynamic version of these finite-difference equations are used until $(k+1)\Delta t > t_b$, where the time t_b is fixed by the condition

$$y_{n_s,k+1}\frac{\Delta x_{n_s,k+1}}{2\Delta s} + (b_{k+1} - x_{n_s,k+1})\frac{\Delta y_{n_s,k+1}}{2\Delta s} = 0 \ . \tag{86}$$

In order to determine t_b we iterate with Δt, the mesh-size of the grid in the t-direction, until (86) is satisfied. Because the system is first order with respect to t this is easy to implement. Precautions have to be taken that the Δt becomes not too small.

From that time on we have a real moving-boundary value problem with $s_w \leq s \leq L, t \geq t_b$. Two additional unknowns the s-coordinate s_w and the jump in the normal force at the moving boundary are introduced. We assume that $i\Delta s \leq s_w \leq (i+1)\Delta s$, where i is an integer such that $0 < i \leq n_s - 1$. For $t_b \leq t$ there are two sets of difference equations, defined on $s \in [L_0, s_w]$, $0 \leq j \leq i$ and $s \in [s_w, L]$, $i+1 \leq j \leq n_s$, where the variable T in the equations for $0 \leq j \leq i$ is replaced by S for $i+1 \leq j \leq n_s$. In the equations of motion (42), (43) and (59), (60) the second order partial derivative of the functions x, y and M with respect to the spatial coordinate s are the highest ones. With the discretization process these functions, as well as their first order partial derivatives, are assumed to be continuous. This means that the conditions (64), (65) and (66) are automatically satisfied if we use the values in the gridpoints i and $i+1$ at both sides of the boundary as if there would be no boundary at all. In point $i+1/2$ we have two unknowns for the normal force $T_{i+1/2,k+1}$ and $S_{i+1/2,k+1}$.

Equations (81) and (82) become for $t_b \leq t$

$$\begin{aligned}(\frac{2}{3}m_s + m_t)\frac{u_{n_s,k+1} - u_{n_s,k}}{\Delta t} &= \\ \mu\left(-\frac{1}{2}(S_{n_s+1/2,k+1} + S_{n_s-1/2,k+1})\frac{\Delta x_{n_s,k+1/2}}{2\Delta s} - \frac{\Delta M_{n_s,k+1}\Delta y_{n_s,k+1}}{4\Delta s^2}\right) &+ \\ (1-\mu)\left(-\frac{1}{2}(S_{n_s+1/2,k} + S_{n_s-1/2,k})\frac{\Delta x_{n_s,k}}{2\Delta s} - \frac{\Delta M_{n_s,k}\Delta y_{n_s,k}}{4\Delta s^2}\right) &, \end{aligned} \tag{87}$$

$$l_{\cdot,k} = L - s_{w_{k+1}} + \left((b_{\cdot,k+1} - x_{s_w,k+1})^2 + y^2_{s_w,k+1}\right)^{1/2} , \tag{88}$$

The equation of motion of the arrow for $t_b \leq t$, (84) and (85) become

$$\frac{b_{k+1} - b_k}{\Delta t} = \mu c_{k+1} + (1-\mu)c_k , \tag{89}$$

$$\begin{aligned}(m_a + \frac{1}{3}m_s)\frac{c_{k+1} - c_k}{\Delta t} = \mu\big(-K_{\cdot,k+1}(b_{k+1} - x_{s_w,k+1})/l_{\cdot,k+1} - (L - s_{w_{k+1}})\big) &+ \\ (1-\mu)\big(-K_{\cdot,k}(b_k - x_{s_w,k})/l_{\cdot,k} - (L - s_{w_k})\big) &. \end{aligned} \tag{90}$$

The condition (68) at the moving boundary $s = s_w$ and for $t_b \leq t$, is discretized using

$$x_{s_w,k+1} = \frac{(s_{w_{k+1}} - i\Delta s)x_{i+1,k+1} - (s_{w_{k+1}} - (i+1)\Delta s)x_{i,k+1}}{\Delta s} , \tag{91}$$

and

$$\begin{aligned}x'_{s_w,k+1} = \big[(s_{w_{k+1}} - i\Delta s)x_{i+2,k+1} - (s_{w_{k+1}} - (i+1)\Delta s)x_{i+1,k+1} &- \\ \big((s_{w_{k+1}} - i\Delta s)x_{i,k+1} - (s_{w_{k+1}} - (i+1)\Delta s)x_{i-1,k+1}\big)\big]/(2\Delta s^2) &, \end{aligned} \tag{92}$$

with similar formulae for $y_{s_w,k+1}$ and $y'_{s_w,k+1}$. These formulae are obtained using the Taylor expansion centred on the moving boundary.

The use of this set of equations for the moving boundary ensures that the equations are, except for the jump in the normal force, continuous with respect to the variable s_w with the transition of one interval into an adjacent one. This holds also with respect to the transition over point of time t_b, when the boundary conditions (49) and (50) at $s = L$ change "smoothly" into (62) and (63), respectively, because of (68).

Finally, we pay attention to the implementation of the proposed method in a computer code. At each time level k+1 the set of nonlinear equations is solved using a modified Newton method, in which the set of linear equations in each iteration step is solved using a Gaussian elimination technique with partial pivoting. It is of profound advantage with respect to computer time and memory storage, that the so called Jacobian matrix possesses a band-structure. After careful inspecting the equations one notices the possibility to renumber the unknowns so that the matrix keeps its band-structure also for $t \geq t_b$. For $t \leq t_b$ the starting values at time level $k+1$ for the Newtonian method are obtained using an extrapolation of the solution of the preceding time level k. For $t_b \leq t$ an initial guess for i is obtained after this extrapolation using the discretizations of equation (68). If after the iteration $s_{w_{k+1}}$ is not in the interval $[i\Delta s, (i+1)\Delta s]$ then the iteration is repeated with the corrected i. This in order to minimize the discretization error.

In the next sections the results obtained with the proposed finite-difference scheme are presented. In order to get a stable solution the parameter m was set equal to 1 for $t \geq t_b$, so for the working recurve bow the fully implicit technique was used.

5 Results and conclusions

The classification of the bow we use, is based on the geometrical shape and the elastic properties of the limbs. The bows of which the upper half is depicted in Figure 3 are called non-recurve bows. These bows have contact with the string only at their tips. In the case of the static-recurve bow, see Figure 4, the outermost parts of the recurved limbs (the ears) are stiff.

In Kooi and Sparenberg [9] we dealt with the statics of the three types of bow. In Kooi [5] we considered the dynamics of the non-recurve bow and in Kooi [6] the dynamics of the static-recurve bow. In this paper we fill this gap of the research into the mechanics of bows and consider the dynamics of the working-recurve bow.

In our mathematical model the action of a bow and arrow combination is fixed by one point in a high dimensional parameter space. Representations of different types of bow used in the past and in our time form clusters in this parameter space. In this section we consider different types of bow: two non-recurve bows, the flat straight-end bow and the Angular bow, two Asian types of static-recurve bow and two working-recurve bows, one with an extreme recurve and a modern working-recurve bow. The precise definition of these bows is given in Kooi [8].

We start with a straight-end bow described by Klopsteg in Hickman [3]. This bow is referred to as the KL-bow. The shape of the KL-bow for various draw-lengths is shown in Figure 3(a). The AN-bow represents the Angular bow found in Egypt and Assyria.

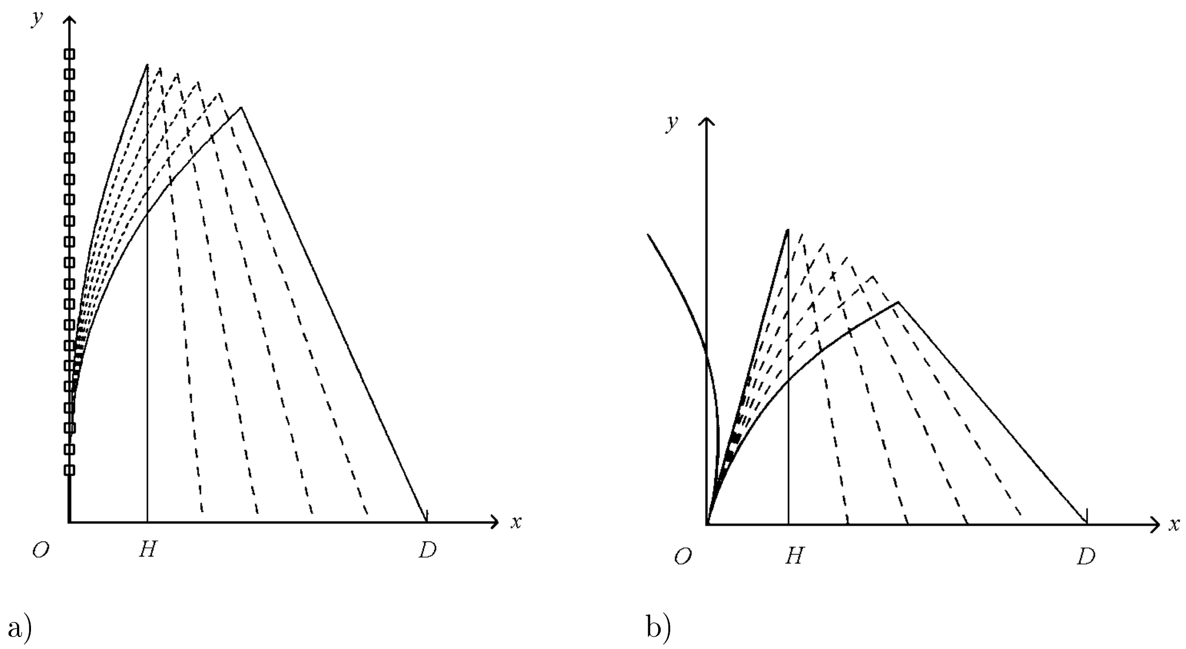


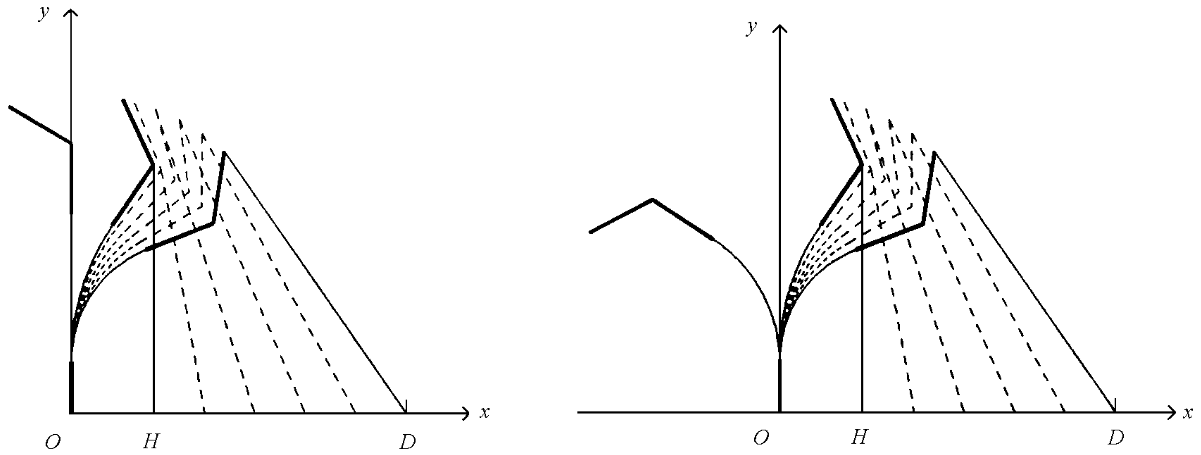
Figure 3: Static deformation shapes (a) of the KL-bow and (b) of the AN-bow.

The shape of the unstrung bow, shown in Figure 3(b), implies that in the braced situation the limb and the string form the characteristic triangular shape. We consider two static-recurve bows, one from China, India and Persia, to be called the PE-bow, and one from Turkey, to be called the TU bow. The shapes of these bows for some draw-lengths are shown in Figure 4. One of a working-recurve bows we consider possesses an excessive recurve to be called the ER-bow. It resembles a bow described and shot by Hickman, see also Hickman [3].

All the quality coefficients for these types of bow are shown in Table 1. The results indicate that the muzzle velocity is about the same for all types. The efficiency of strongly recurved bows is rather bad. So, within certain limits, these dimensionless parameters appear to be less important than is often claimed.

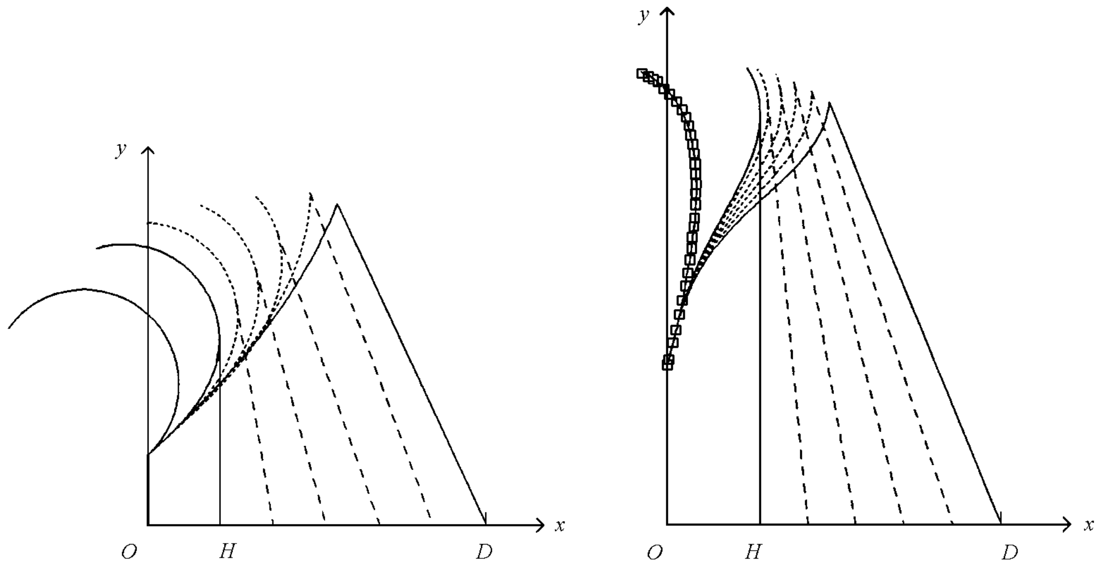
In Table 1 the values of the quality coefficients for a modern working-recurve bow are also given. Also these coefficients differ not much from the other values. In Kooi [7] it is shown that the materials used for modern working-recurve bows can store more deformation energy per unit of mass than the materials used in the past. Hence, this contributes most to the improvement of the bow.

The shape of the bow for a number of draw-lengths is shown in Figure 5. Figure 6 gives the shape of the limb and the string, 6(a) before $(0 \leq t \leq t_l)$ and 6(b) after $(t_l \leq t)$ arrow exit. Observe the large vibrations of the string after the arrow leaves the string, which imply that the brace height has to be large. In Figure 7 the Static Force Draw (SFD) curve $F(b)$ and the Dynamic Force Draw (DFD) curve $E(b)$ for the modern working-recurve bow are compared.



a) b)

Figure 4: Static deformation shapes (a) of the PE-bow and (b) of the TU-bow.



a) b)

Figure 5: Static deformation shapes (a) of the ER-bow and (b) of the modern working-recurve bow WR-bow.

Table 1: Dimensionless quality coefficients for a number of bows.

Bow	q	η	ν	m_a	m_s	$W(L_0)$	$V(L_0)$
KL-bow	.407	.765	2.01	.0769	.0209	1.4090	1.575
AN-bow	.395	.716	1.92	.0769	.0209	0.2385	2.300
PE-bow	.432	.668	1.94	.0769	.0209	0.2304	1.867
TU-bow	.491	.619	1.99	.0769	.0209	0.1259	1.867
ER-bow	.810	.417	2.08	.0769	.0209	0.3015	2.120
WR-bow	.434	.729	2.23	.0629	.0222	2.5800	1.950

The acceleration force E is defined by

$$E = -2m_a\dot{c}\ , \quad t \geq 0\ . \tag{93}$$

These results show that the modern working-recurve bow is a good compromise between the non-recurve bow and the static-recurve bow. The recurve yields a good static quality coefficient and the light tips of the limbs give a reasonable efficiency. This calculated efficiency of the modern working-recurve bow correlates well with values given in the literature.

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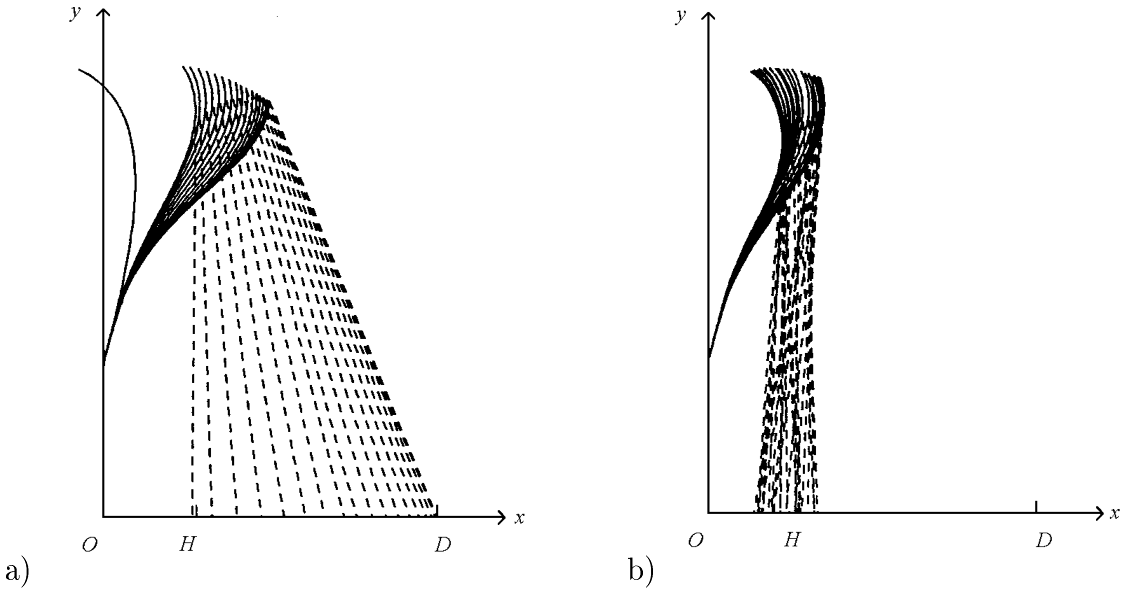

Figure 6: Dynamic deformation shapes of the WR-bow (a) for $0 \leq t \leq t_l$ and (b) for $t_l \leq t$.

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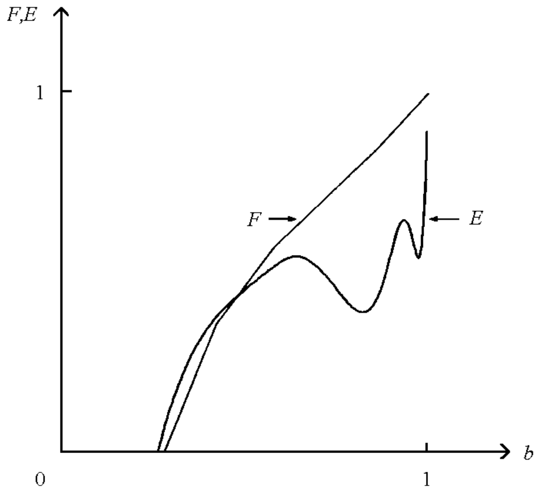

Figure 7: Static Force Draw curve and Dynamic Force Draw curve for the WR-bow.